SINGULAR POINT PROBLEMS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS†

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1. **Introduction.** The discussion of the subject indicated in the title is not intended in any way to be encyclopaedic. The object of this address relates to the general problems of determining the character of solutions of equations

$$\text{(A)}\quad L_n(y) \equiv a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y^{(1)} + a_n(x)y = 0,$$

$$\text{(B)}\quad L_n(x, \lambda; y) \equiv a_0(x, \lambda)y^{(n)} + a_1(x, \lambda)y^{(n-1)} + \cdots + a_n(x, \lambda)y = 0.$$

The $a_i(x)$ in (A) are assumed to be analytic for $|x| \geqq \rho$, $(x \neq \infty)$, being representable by convergent series of the form

$$\text{(1.1)}\qquad \alpha_i(x) = x^{n_i/p}[a_{i,0} + a_{i,1}x^{-1/p} + a_{i,2}x^{-2/p} + \cdots],$$

$$n_i,\ p \text{ integers};\ p > 0;\ i = 0, \cdots, n;\ |x| \geqq \rho,$$

or they are supposed to be merely asymptotic to such possibly divergent series, for x in a suitable region extending to infinity. The $a_i(x, \lambda)$ in (B) are assumed to be indefinitely differentiable in x, for x on a closed real interval (a, b), and analytic in λ for $|\lambda| \geqq \rho > 0$, $(\lambda \neq \infty$; λ is a parameter), being representable by convergent series of the form

$$\text{(1.2)}\qquad \alpha_i(x, \lambda) = \lambda^{n_i}\sum_{\nu=0}^{\infty} \alpha_{i,\nu}(x)\lambda^{-\nu}, \qquad n_i \text{ integers};\ i = 0, \cdots, n;$$

$$\alpha_{i,\nu}(x) \text{ indefinitely differentiable on } (a, b);\ a \leqq x \leqq b;\ |\lambda| \geqq \rho,$$

or they are supposed to be asymptotic to a finite number of terms, or to infinitely many terms, to such possibly divergent series, when x is on (a, b) and the parameter λ is in a suitable region R extending to infinity.

The investigation for the problems (A) and (B) (relating to equations (A) and (B), respectively) has the purpose of establishing the character of solutions in the complex neighborhood of the singular point of the equation under consideration. In problem (A) the singular point is at $x = \infty$. In problem (B) the singular point considered is at $\lambda = \infty$, (x in (a, b)).

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† An address delivered by invitation of the Program Committee at the Iowa City Meeting of the Society, November 26, 1937.

These problems are important, in part, because many significant classical instances of differential equations occur as particular cases of (A) and (B); on the other hand, a satisfactory theory of problems (A) and (B) is a very desirable prerequisite for the analytic theories of certain highly important non-linear differential systems.

We use the term *asymptotic relation* in the following sense. If R is a region extending to infinity in the complex x plane it will be said that a function $a(x)$, defined in R, is asymptotic to $\alpha(x) = a_0 + a_1 x^{-1/p} + \cdots$ for x in R (that is, $a(x) \sim \alpha(x)$ in R) if

$$(1.3) \qquad a(x) = a_0 + \cdots + a_{m-1}x^{-(m-1)/p} + \beta_m(x)x^{-m/p}, \qquad |\beta_m(x)| \leqq \beta_m \text{ in } R;\ m = 1, 2, \cdots.$$

If R is a region extending to infinity in the complex λ plane, we shall say that a function $a(x, \lambda)$, defined for x in (a, b) and λ in R, is asymptotic to $\alpha(x, \lambda) = \alpha_0(x) + \alpha_1(x)\lambda^{-1/p} + \cdots$, (integer $p > 0$), in R (with x on (a, b)) if the $\alpha_r(x)$ are continuous on (a, b) (which interval is taken closed) and

$$(1.4) \qquad a(x, \lambda) = \alpha_0(x) + \cdots + \alpha_{m-1}(x)\lambda^{-(m-1)/p} + \beta_m(x, \lambda)\lambda^{-m/p},$$

$$|\beta_m(x, \lambda)| \leqq \beta_m; \beta_m \text{ independent of } x, \lambda; m = 1, 2, \cdots; x \text{ in } (a, b); \lambda \text{ in } R.$$

Unless the contrary is stated the designation

$$(1.5) \qquad a(x) \sim \alpha(x) \text{ in } R, \qquad a(x, \lambda) \sim \alpha(x, \lambda), \qquad x \text{ in } (a, b);\ \lambda \text{ in } R,$$

will indicate asymptotic relations in the sense that (1.3), (1.4) hold in the respective cases. Such asymptotic relations will be termed *ordinary* or *to infinitely many terms*. When (1.3) (or (1.4)) is asserted only for $m = 1, 2, \cdots, m_0$, then we shall write

$$(1.5a) \qquad a(x) \underset{m_0}{\sim} \alpha(x) \text{ in } R \quad (\text{or } a(x, \lambda) \underset{m_0}{\sim} \alpha(x, \lambda),\ x \text{ in } (a, b);\ \lambda \text{ in } R),$$

the symbol $\underset{m_0}{\sim}$ denoting an asymptotic relation "to m_0 terms."

In consequence of a paper by E. Fabry† it can be asserted that the formal *equation* (A*) *obtained from* (A) *by replacing the* $a_i(x)$ *by the corresponding series* $\alpha_i(x)$ (of (1.1))‡ has n formally linearly independent solutions

$$(1.6) \qquad s_i(x) = e^{Q_i(x)}x^{r_i}\sigma_i(x), \qquad i = 1, \cdots, n,$$

where the $Q_i(x)$ are polynomials in x^{1/k_i} and

† E. Fabry, *Sur les intégrales des équations différentielles linéaires à coefficients rationnels*, Thèse, 1885, Paris.

‡ It is assumed that in $\alpha_0(x)$ not all the coefficients are zero. This hypothesis is made for convenience and has no essential significance.

$$(1.6a)\qquad \sigma_i(x) = \sigma_i^0(x) + \sigma_i^1(x)\log x + \cdots + \sigma_i^m(x)\log^{m_i} x,$$

$$(1.6b)\qquad \sigma_i^\gamma(x) = \sigma_i^{\gamma,0} + \sigma_i^{\gamma,1}x^{-1/k_i} + \cdots,$$

$$\gamma = 0, \cdots, m_i;\ i = 1, \cdots, n;$$

here m_i, k_i are integers ($m_i \geqq 0$; $k_i(>0)$ is an integral multiple of p), and the series (1.6b) may diverge (for all x).

In consequence of the work of P. Noaillon† the formal equation (B*), obtained from (B) by replacing the $a_i(x, \lambda)$ by the corresponding series $\alpha_i(x, \lambda)$ (of (1.2)), possesses a full set of n formal (in general divergent) series solutions

$$(1.7)\qquad s_i(x, \lambda) = e^{Q_i(x,\lambda)}\sigma_i(x, \lambda), \qquad i = 1, \cdots, n,$$

where the $Q_i(x, \lambda)$ are polynomials in λ^{1/k_i} (integer $k_i > 0$) and

$$(1.7a)\qquad \sigma_i(x, \lambda) = \sum_{\nu=0}^{\infty} \sigma_{i,\nu}(x)\lambda^{-\nu/k_i}, \qquad i = 1, \cdots, n,$$

the coefficients in $Q_i(x, \lambda)$ and in the series $\sigma_i(x, \lambda)$ being functions indefinitely differentiable with respect to x for x in a suitable interval (a, b).‡

The essential features of an analytic theory of equations (A) and (B) relate to the following questions.

I. If the $a_i(x)$ (or $a_i(x, \lambda)$) are representable by convergent series $\alpha_i(x)$ of (1.1) (or series $\alpha_i(x, \lambda)$ of (1.2)), under what conditions do the formal series solutions (all or any) converge, thus representing "actual" solutions of (A) (or of (B))?

II. If the $a_i(x)$ (or $a_i(x, \lambda)$) are representable by convergent series of the type stated above, under what conditions are the formal solutions, all or any, (1.6) (or (1.7)), though possibly divergent, summable to "actual" solutions?

With respect to II a method which is particularly potent is that involving what essentially amounts to exponential summability—to be precise, Laplace integrals leading to expressions containing convergent factorial series.§

† P. Noaillon, *Développements asymptotiques dans les équations différentielles linéaires à paramètre variable*, Mémoires de la Société des Sciences de Liège, (3), vol. 9 (1912), 197 pages.

‡ Possibly merely a sub-interval of the interval originally so denoted.

§ Developments of this type rest on important results obtained by N. E. Nörlund in the theory of factorial series (Nörlund applies these methods to difference equations); cf. N. E. Nörlund, Acta Mathematica, vol. 37 (1914), pp. 327–387; *Leçons sur les Équations Linéaires aux Différences Finies*, Paris, 1929. References to J. Horn's applications of methods mentioned in II to differential equations will be given later.

III. If the $a_i(x)$ (or $a_i(x, \lambda)$) are representable as specified in I and II, but the methods suggested therein (that is, convergence and summability) fail, what can be done, in so far as "actual" solutions are concerned, by asymptotic methods? When the $a_i(x)$ (or $a_i(x, \lambda)$) are merely asymptotic to series of the type specified above, questions I, II lose their significance and only question III remains.

This address relates mainly to my own work regarding questions II *and* III.† *Of my three papers, just referred to,* (T_1) *and* (T_2) *relate to question* III *for the problems* (A) *and* (B), *respectively.* (T_3) *relates to question* II *for the problem* (A).‡ From a certain point of view (T_1), (T_2), (T_3) give a complete treatment of certain essential aspects of the theory of equations (A) and (B). The most relevant predecessors to these papers are certain contributions of a number of mathematicians. Of the predecessors to (T_1) (relating to question III for the problem (A)), highly significant are papers of H. Poincaré§ and G. D. Birkhoff.‖ With respect to (T_2) (relating to question III for the problem (B)) should be mentioned important developments due to G. D. Birkhoff,¶ as well as certain contributions due to Noaillon (already referred to) and to J. D. Tamarkin.** In connection with (T_3) (relating to question II for (A)) of the relevant predecessors we shall mention J. Horn.††

2. Problem (A) **(asymptotic methods).** H. Poincaré treats equation

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† W. J. Trjitzinsky, *Analytic theory of linear differential equations*, Acta Mathematica, vol. 62 (1934), pp. 167–226. Referred to as (T_1).

W. J. Trjitzinsky, *Theory of linear differential equations containing a parameter*, Acta Mathematica, vol. 67 (1936), pp. 1–50. Referred to as (T_2).

W. J. Trjitzinsky, *Laplace integrals and factorial series in the theory of linear differential and linear difference equations*, Transactions of this Society, vol. 37 (1935), pp. 80–146. Referred to as (T_3).

‡ For the problem (B) summability methods, on the whole, appear to be rather ineffective.

§ H. Poincaré, American Journal of Mathematics, vol. 7 (1885), pp. 203–258.

‖ G. D. Birkhoff, Transactions of this Society, vol. 10 (1909), pp. 436–470.

¶ G. D. Birkhoff, *On the asymptotic character of solutions of certain linear differential equations*, Transactions of this Society, vol. 9 (1908), pp. 219–231; also cf. ibid., vol. 9 (1908), pp. 373–395; also see G. D. Birkhoff and R. E. Langer, *The boundary problems and developments . . .*, Proceedings of the American Academy of Arts and Sciences, vol. 58 (1923), pp. 51–128.

** J. D. Tamarkin, *Some general problems of the theory of ordinary linear differential equations . . .*, Mathematische Zeitschrift, vol. 27 (1927), pp. 1–54.

†† Some of Horn's work is as follows: *Integration linearer Differentialgleichungen . . .*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 24 (1915), pp. 309–329; *Laplacesche Integrale . . .*, Mathematische Zeitschrift, vol. 21 (1924), pp. 85–95.

(A), (§1) of rank one, with the roots of the characteristic equation distinct.† G. D. Birkhoff's work relates to equation (A), (§1) of any rank, the roots of the characteristic equation being distinct.‡ The methods of these two writers are not readily extensible to the general problem (A).

In (T_1), *with the aid of "iterations," factorizations, and appropriate integrations, is established a fundamental existence theorem for the general problem* (A). To state this theorem conveniently a number of definitions will be introduced.

DEFINITION 1. Generically $\{x\}_q$, (q an integer $\geqq 0$), will denote an expression

$$\rho_0(x) + \rho_1(x) \log x + \cdots + \rho_q(x)\log^q x,$$

where the $\rho_j(x)$ are series, possibly divergent, of the form

$$\rho_{j,0} + \rho_{j,1}x^{-1/k} + \rho_{j,2}x^{-2/k} + \cdots, \qquad k \text{ a positive integer.} \tag{2.1}$$

DEFINITION 2. A curve B will be said to be *regular* if it is simple and extends to infinity where it possesses a unique limiting direction.§

DEFINITION 3. A region R will be said to be *regular* if it is closed, extends to infinity, and is such that if x is in R, then $|x| \geqq r_1 > 0$; also the boundary of R is to be simple and is to consist of an arc γ of the circle $|x| = r_1$ and of two regular curves extending from different extremities of γ. Generically $R(\theta_1, \theta_2)$ will designate a regular region for which the two regular curves (forming part of the boundary) have limiting directions θ_1 and θ_2, respectively. The number $|\theta_1 - \theta_2|$ is to be termed the opening of R.

DEFINITION 4. The symbol $[x]_q$ will denote a function of the form $p_0(x) + \cdots + p_q(x) \log^q x$ where the $p_j(x)$ are functions, analytic for $x \neq \infty$ in a regular region R, such that

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† When (A) is written in a suitable form one then has the $Q_i(x)$ in (1.6) of the form $h_i x$, no logarithms will be present in (1.6a), and the series (1.6b) will be in powers of x^{-1}.

‡ With (A) taken in a suitable form, the $Q_i(x)$ of (1.6) will be polynomials in x (of degree which may be higher than the first), no logarithms will enter in (1.6a), and the series (1.6b) will be in powers of x^{-1}. The coefficients of the leading powers in the $Q_i(x)$ are expressible in terms of the roots of the characteristic equation; the latter is of the form $E(g) = 0$ where $E(g)$ is a polynomial of degree n, the coefficients of the various powers of g being certain ones of the initial coefficients from the series (1.1).

§ That is, if the equation of the curve is $\theta = f(r)$, (θ = angle of x; $r = |x|$), then $\lim \theta$ (as $r \to \infty$) exists and is unique.

$$p_j(x) \sim \rho_j(x) \text{ in } R,$$

the $\rho_j(x)$ being series of the form (2.1).

One has $[x]_q \sim \{x\}_q$, x in R.†

It is a well known fact that when the coefficients in (A) are representable by convergent series (1.1), while in the formal solutions the polynomials $Q_i(x)$, $(i=1, \cdots, n)$, are *all* identically zero, the formal series (1.6) will *all* converge (for $|x| \geqq r'$; r' sufficiently great) and will thus represent a full set of "actual" solutions of (A). The equation (A) is then said to be of Fuchsian type.

We let $B_{i,j}$ denote a regular curve along which

$$R(Q_i(x) - Q_j(x)) = 0. \tag{2.2}$$

Such curves will be defined only when $Q_i(x) \not\equiv Q_j(x)$. Let

$$R_1, R_2, \cdots, R_N \tag{2.3}$$

be regular regions separated by the $B_{i,j}$ curves (formed for all admissible pairs of values i, j) in such a way that interior to any such region there is no $B_{i,j}$ curve. Consider a particular region R_k of the set (2.3). It has the form $R(\theta_{k,1}, \theta_{k,2})$, where $\theta_{k,1} \leqq \theta_{k,2}$. The regular curves forming part of the boundary of R_k and possessing at infinity the limiting directions $\theta_{k,1}$ and $\theta_{k,2}$ will be designated as $B_{k,1}$ and $B_{k,r}$, respectively.

In view of the fundamental existence theorem given in (T_1) the following can be asserted for any fixed k, $(1 \leqq k \leqq N)$.

If $\theta_{k,1} = \theta_{k,2}$, the equation (A) *has a full set of solutions*

$$y_i(x), \qquad i = 1, \cdots, n,$$

whose elements $y_i(x)$ are analytic in R_k $(x \neq \infty)$, while

$$y_i(x) \sim s_i(x), \text{ in } R_k; \qquad i = 1, \cdots, n, \tag{2.4}$$

the series $s_i(x)$ being given by (1.6); *that is,*

$$y_i(x) = e^{Q_i(x)} x^{r_i} [x]_{q_i}, \text{ in } R_k; \qquad i = 1, \cdots, n.$$

If $\theta_{k,1} < \theta_{k,2}$, there exist regular overlapping subregions of R_k,

$$_rR_k = R(\theta_{k,1}, \theta_{k,2}), \qquad {}_lR_k = R(\theta_{k,1}, \theta_{k,2}), \tag{2.5}$$

whose boundaries contain $B_{k,l}$ and $B'_{k,r}$, respectively.‡ So that there exist two full sets of "actual" solutions,

† In the sense indicated in the definitions.

‡ The other regular curve, which forms part of the boundary of $_rR_k$ (or $_lR_k$), is interior to R_k and has at infinity the limiting direction of $B'_{k,r}$ (or $B'_{k,l}$). The precise details are given in (T_1).

(2.5a) $\quad {}_rY_i(x), \quad (i = 1, \cdots, n); \quad {}_lY_i(x), \quad i = 1, \cdots, n,$

such that

(2.5b) $\quad {}_rY_i(x) \sim s_i(x), \quad i = 1, \cdots, n;$ *in* ${}_rR_k,$

(2.5c) $\quad {}_lY_i(x) \sim s_i(x), \quad i = 1, \cdots, n;$ *in* ${}_lR_k;$

that is, the functions (2.5a) *are of the form* $(\exp Q_i(x))x^{r_i}[x]_{q_i}$, $(i=1, \cdots, n)$, *in the specified regions.*

Results of the above description can be stated for each one of the regions (2.3). If we take a particular set of solutions relating, let us say, to R_1, the asymptotic behaviour of these solutions can be studied in the complete neighborhood of $x = \infty$ (for details, cf. (T_1)).†

Under certain conditions results of the stated kind will hold in more extensive regions (see (T_1), pp. 216–218). However, *it is not to be expected that in general one could replace* ${}_rR_k$ *and* ${}_lR_k$ *(the regions involved in* (2.5b) *and* (2.5c)) *by their sum* R_k *and assert existence of a full set of "actual" solutions* $y_i(x)$, $(i=1, \cdots, n)$, *for which* $y_i(x) \sim s_i(x)$, $(i=1, \cdots, n)$, *throughout* R_k. In other words, no essential improvement is to be expected in this direction. The paper (T_1) discusses also some converse problems, including the formulation of the corresponding Riemann problem.

3. **Problem** (B) **(asymptotic methods).** It is convenient to write equation (B) of §1 in the form

$$(B') \qquad L(x, \lambda; y) \equiv \sum_{k=0}^{n} \lambda^{H(n-k)} a_{n-k}(x, \lambda) y^{(k)} = 0, \qquad \text{integer } H \geqq 0,$$

where the $a_{n-k}(x, \lambda)$ are either equal to convergent series of the form

$$(3.1) \qquad \alpha_{n-k}(x, \lambda) = \sum_{\nu=0}^{\infty} \alpha_{n-k,\nu}(x)\lambda^{-\nu}, \qquad k = 0, 1, \cdots, n,$$

or are asymptotic (as stated subsequent to (1.2)) to such possibly divergent series, respectively.‡ We choose the integer $H(\geqq 0)$ as small as possible. In §1 it has been assumed that in the series $\alpha_0(x, \lambda)$ not all the coefficients are identically zero. Hence, with (a, b) properly chosen, we now may suppose that $\alpha_0(x, \lambda) \equiv 1$.

If $H = 0$, while $a_{n-k}(x, \lambda) = \alpha_{n-k}(x, \lambda)$ (the latter series being convergent for x in (a, b) and for $|\lambda| \geqq r' > 0$), we have an analog to

† In general, of course, there is change of asymptotic form from curve to curve, while in the neighborhood of some of the curves the asymptotic form, figuratively speaking, may become "blurred."

‡ The $\alpha_{n-k,\nu}(x)$ are of the same description as the $\alpha_{i,\nu}(x)$ in (1.2).

the Fuchsian theory. As established by H. Poincaré there exists then a full set of solutions analytic in λ at $\lambda = \infty$ (provided the initial conditions are of the same character).

The characteristic equation for (B′) is

$$E(x; g) \equiv \sum_{k=0}^{n} \alpha_{n-k,0}(x) g^k = 0. \tag{3.2}$$

Let $g_i = g_i(x)$, $(i = 1, \cdots, n)$, be its roots. The interval (a, b) will be so chosen that the following will be true for any pair of functions $g_i(x)$, $g_j(x)$, $(i \neq j)$. *Either* $g_i(x) = g_j(x)$ *for all* x *in* (a, b) *or* $g_i(x) \neq g_j(x)$ *for every* x *in* (a, b).† With a proper choice of (a, b), then, (B′)‡ will possess a full set of formal solutions (1.7). The latter involve functions $Q_i(x, \lambda)$, $(i = 1, \cdots, n)$. Now, $Q_i(x, \lambda)$ *will be a polynomial in* λ^{1/k_i}, (*integer* $k_i > 0$). With the highest power of λ displayed, one may write

$$Q_i(x, \lambda) = q_{i,0}(x)\lambda^H + \cdots; \tag{3.3}$$

here

$$q^{(1)}_{i,0}(x) \equiv g_i(x), \qquad i = 1, 2, \cdots, n, \tag{3.3a}$$

where $g_i(x)$ is a root of (3.2).

The number H has to do with what appropriately might be called the rank of the equation (B). Whence H *will be termed the rank of* (B).

G. D. Birkhoff gave an asymptotic theory, with the essential particulars, for the equations (B) *of rank unity in the case when the roots of the characteristic equation* (3.2) *are distinct throughout* (a, b).

J. D. Tamarkin developed a theory for equations (B) *of any finite rank under the assumption that the roots of the characteristic equation* (3.2) *are distinct in* (a, b).

The extension of the methods of these writers to the treatment of the more general problem, now under consideration, cannot be conveniently carried out.

In (T_2) *we develop the theory of the general problem* (B) (*of any finite rank* H, *admitting existence of multiple roots of* (3.2)). The fundamental existence theorem of (T_2) is established with the aid of a limiting process called "iterations" akin to an analogous process which we introduced in (T_1), suitable integrations, and factorizations.

The existence theorem of (T_2) asserts in particular the following:

† (a, b) is closed.

‡ Rather, the equation (B′*) obtained by replacing the $a_{n-k}(x, \lambda)$ in (B′) by the series $\alpha_{n-k}(x, \lambda)$, respectively.

Let the $B_{ij}^{1,x}$ designate curves in the complex λ-plane, extending to infinity, along which

$$RQ_i^{(1)}(x, \lambda) = RQ_j^{(1)}(x, \lambda). \dagger \tag{3.4}$$

There exist subintervals of (a, b) such that, when x is restricted to one of them, there exists a regular region R in the λ-plane so that there are no curves $B_{ij}^{1,x}$, $(i, j=1, \cdots, n)$, interior to R. Let (a', b') denote any particular subinterval of the kind referred to above. As a matter of notation, entailing no loss of generality, one may write

$$RQ_1^{(1)}(x, \lambda) \leqq RQ_2^{(1)}(x, \lambda) \leqq \cdots \leqq RQ_n^{(1)}(x, \lambda), \qquad x \text{ in } (a', b'); \lambda \text{ in } R.$$

Let α be an integer, however large. The equation (B) *will possess a full set of "actual" solutions, ${}_\alpha y_i(\lambda, x)$, $(i=1, \cdots, n)$, with elements analytic in λ, $(\lambda \neq \infty)$, and continuous in x for x in (a', b') and λ in R; moreover,*

$${}_\alpha y_i(\lambda, x) \sim s_i(\lambda, x), \qquad i = 1, \cdots, n; x \text{ in } (a', b'); \lambda \text{ in } R, \tag{3.5}$$

where the $s_i(\lambda, x)$ are the series of (1.7).

For the problem under consideration *regions R and full sets of "actual" solutions, of the above description, will certainly exist at least for some sufficiently small interval (a', b').*

Associated with the equation (B′) is the system, which in matrix notation may be written as

$$Y^{(1)}(x, \lambda) = Y(x, \lambda)D(x, \lambda), \qquad Y(x, \lambda) = (y_{i,j}(x, \lambda)), \ddagger \tag{B″}$$

where

$$D(x, \lambda) = \begin{bmatrix} 0, & 0, & \cdots, & -\lambda^{H_n} a_n(x, \lambda) \\ 1, & 0, & \cdots, & -\lambda^{H_{(n-1)}} a_{n-1}(x, \lambda) \\ \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & \cdots, & -\lambda^{H} a_1(x, \lambda) \end{bmatrix} = (d_{i,j}(x, \lambda)).$$

If $(v_{i,j}(x, \lambda))$ is a matrix solution of (B′), then $(y_{i,j}(x, \lambda))$

† Differentiation is with respect to x. Whenever $Q_i(x,\lambda) \equiv Q_j(x,\lambda)$ the convention will be that there are no corresponding $B_{ij}^{1,x}$ curves. Sufficiently far from the origin the $B_{ij}^{1,x}$ curves are simple; moreover, at infinity they will possess limiting directions. A particular curve (3.4) will depend on x. As x varies in the interval this curve may vary; the angle of the sector within which this variation takes place can be made as small as desired by a suitable choice of the interval for x.

‡ $(y_{i,j}(x, \lambda))$ is a matrix of n^2 elements, with $y_{i,j}(x, \lambda)$ in the ith row and jth column $(i, j=1, \cdots, n)$.

$=(y_{i,1}^{(j-1)}(x, \lambda))$, and the $y_{i,1}(x, \lambda)$, $(i=1, \cdots, n)$, will constitute a full set of solutions of (B′). If the $y_i(x, \lambda)$ $(=y_{i,1}(x, \lambda); i=1, \cdots, n)$ form a full set of solutions of (B′), the matrix $Y(x, \lambda)=(y_i^{(j-1)}(x, \lambda))$ will satisfy (B″). More generally, there exists a type of reciprocal relationship between a matrix system

$$(B''') \qquad Y^{(1)}(x, \lambda) = Y(x, \lambda)A(x, \lambda), \qquad A(x, \lambda) = (a_{i,j}(x, \lambda)),$$

where the $a_{i,j}(x, \lambda)$, $(i, j=1, \cdots, n)$, are functions of the same type as the $a_i(x, \lambda)$ in (B), (§1),† and a single equation (B). The system (B‴) is satisfied by a formal matrix $S(x, \lambda)=(s_{i,j}(x, \lambda))=(\sigma_{i,j}(x, \lambda) \cdot \exp Q_i(x, \lambda))$, where the series $\sigma_{i,j}(x, \lambda)$, $(i, j=1, \cdots, n)$, are of the same description as the $\sigma_i(x, \lambda)$ of (1.7a). The elements of a row will constitute a formal solution.

Under conditions and with the notation similar to that under which (3.5) has been asserted the following can be stated:

There exists an "actual" matrix solution of (B‴), ${}_{\alpha}Y(x, \lambda)$, *such that*

$${}_{\alpha}Y(x, y) \underset{\alpha}{\sim} S(x, \lambda), \qquad x \text{ in } (a', b'); \lambda \text{ in } R.$$

Such a matrix can be constructed for $\alpha=1, 2, \cdots$.

Any matrix $Z(x, \lambda)={}_{\alpha}Y^{-1}(a', \lambda)_{\alpha}Y(x, \lambda)$ (or ${}_{\alpha}Y^{-1}(b', \lambda)_{\alpha}Y(x, \lambda)$) will be independent of α and will, of course, satisfy the system (B‴). As pointed out in (T_2), $Z(x, \lambda)$ *will satisfy certain asymptotic relations in the ordinary sense*‡ *for* x *on* (a', b') *and* λ *in* R.

In (T_2), pp. 40–44, are also made applications to integro-differential equations of the form

$$L(x, \lambda; y(x, \lambda)) = a(x, \lambda) + \int_{a'}^{x} b(u, x, \lambda)y(u, \lambda)du,$$

where L is the differential operator of the left member of (B), (§1).

On the basis of our existence results some applications are also made to boundary value problems.§ For the general problem (B), however, development of an adequate boundary value theory (leading to expansions of arbitrary functions) apparently necessitates

† It is assumed that the determinant $|A(x, \lambda)| \neq 0$. This relationship is not obvious but is to be expected and is established without much difficulty.

‡ Cf. (T_2), p. 33.

§ A formulation of general boundary value problems, together with a number of significant results, is given by J. D. Tamarkin, Mathematische Zeitschrift, loc. cit. A treatment of boundary value problems for linear systems, with the parameter entering linearly, is due to Birkhoff and Langer, Proceedings of the American Academy of Arts and Sciences, loc. cit.

some restrictions with respect to the character of the polynomials $Q_i(x, \lambda)$, $(i=1, \cdots, n)$. Thus, it appears, we would have to assume that the various regions for which (B) has solutions of known asymptotic form (as implied by the existence theorem of (T_2)) abut on each other.

4. **Problem** (A) **(Laplace integrals).** The equation (A), (§1), will be now assumed to possess coefficients representable by convergent series of the form (1.1). By means of a transformation $x^\alpha = x_1$ (α suitably chosen) the equation (A) is brought to the form

$$(A') \qquad L(y) \equiv \sum_{k=0}^{n} d_{n-k}(x) y^{(k)} = 0,$$

where we wrote x without the subscript; here†

$$(4.1) \qquad d_{n-k}(x) = \sum_{\nu=0}^{\infty} d_{n-k,\nu} x^{-\nu/p}, \qquad \text{integer } p > 0;\ d_{0,0} \neq 0,$$

all the series being convergent for $|x| \geqq r'(>0)$. The characteristic equation of (A′) will be

$$(4.2) \qquad E(\rho) \equiv \sum_{k=0}^{n} d_{n-k,0} \rho^k = 0.$$

Horn's results,‡ as stated for (A′), are as follows:

Under the hypothesis that the roots of the characteristic equation (4.2) *are distinct,* (A′) *possesses a full set of solutions of the form*

$$e^{Q_i(x)} x^{r_i} \eta_i(x), \qquad i = 1, 2, \cdots, n,$$

where the $Q_i(x)$ *are polynomials in* $x^{1/p}$ *and the* $\eta_i(x)$ *are of the form*

$$\eta_i(x) = \eta_{i,0} + \sum_{\nu=1}^{p} x^{1-\nu/p} \eta_{i,\nu}(x),$$

the $\eta_{i,\nu}(x)$, $(\nu=1, \cdots, p;\ i=1, \cdots, n)$, *being convergent factorial series of the type*

$$\eta_{i,\nu}(x) = \sum_{s=0}^{\infty} \frac{a_s^{i,\nu}}{x(x-\gamma)\cdots(x-s\gamma)}.$$

Here $|\gamma|$ must be sufficiently great and γ is the same for all solutions. Moreover, $\angle\gamma$ is allowed to have any value except certain ones,

† p generally will be different from the integer so denoted in (1.1).

‡ We state them in a form slightly different from that of Horn.

depending on the roots of (4.2).† Formally, solutions (4.2) are compatible with the corresponding formal solutions.

Our treatment of (A′) along these lines, as given in (T_3)‡ goes further. The pertinent results from (T_3) will be forthwith stated.

In accordance with facts pointed out in §1, equation (A′) has a full set of formal solutions $s_i(x)=x^{r_i}\sigma_i(x)\exp Q_i(x)$, $(i=1,\cdots,n)$, where the $\sigma_i(x)$ are of the form (1.6a), and the $Q_i(x)$ are polynomials in x^{1/k_i}, while the coefficients of the powers of $\log x$ in the $\sigma_i(x)$ are power series in x^{-1/k_i} (k_i integers). It will be said that $s_i(x)$ is *normal*§ when $k_i=p$. When $k_i=r_i'p$ (integer $r_i'>1$) the corresponding formal solution $s_i(x)$ will be said to be *anormal*. Consider a root ρ of (4.2) of multiplicity ϕ ($\geqq 1$). If $\phi=1$, the corresponding formal solution will be normal and will contain no powers of $\log x$.¶ If $\phi>1$, the following cases may present themselves:

CASE 1. The ϕ polynomials $Q_i(x)$ from the formal solutions corresponding to the root ρ of (4.2) are not all identical.

CASE 2. This is the alternative to the previous case.

In Case 2 the formal solutions corresponding to the root ρ are all normal.

The theorem (for differential equations) proved in (T_3) can be stated as follows:

Consider a root ρ_1, of multiplicity ϕ, of the characteristic equation $E(\rho)=0$ (see (4.2)) associated with (A′). *Assume that with respect to this root Case 2 takes place. As a matter of convenience, entailing no loss of generality, we may take $\rho_1=0$. For every $\bar{\imath}$, not coincident with a value of an angle of a non-zero root of $E(\rho)=0$, the following will be true.*

(A′) *possesses a set of ϕ (linearly independent) solutions*

$$y_j(x) = e^{Q(x)}x^r\sum_{h=0}^{j-1}\log^h x\left[{}_h\eta_0^{j-1}+\sum_{\nu=1}^{p}x^{(p-\nu)/p}\cdot\sum_{s=0}^{\infty}\frac{{}_hA_{\nu,s}^{j-1}(\bar{\imath})}{x(x-\gamma)\cdots(x-s\gamma)}\right], \tag{4.3}$$

$$j=1,\cdots,\phi;\ Q(x)=\text{polynomial in } x^{1/p},$$

where $|\gamma|$ is suitably great and $\angle\gamma=\bar{\gamma}=-\bar{\imath}$; the factorial series in (4.3) *will all converge in a half-plane*

$$R(\epsilon^{i\bar{\imath}}x)<-q\ (<0).$$

† For a given admissible γ the series (4.2b) will converge in a certain half-plane.

‡ W. J. Trjitzinsky, Transactions of this Society, loc. cit.

§ This is the value p involved in (4.1).

¶ This is the case treated by Horn.

The implications of this theorem for (A), (§1) are obvious. The ϕ corresponding solutions of (A) will be of the form (4.3), with x replaced by a certain power of x; as a consequence the series involved in these solutions will converge in certain sectors.

The above process constitutes, of course, a process of summing the corresponding formal series solutions.

The theorem is of greatest possible completeness in the sense that even normal formal series solutions do not always lead to convergent factorial series developments, if, corresponding to the multiple root in question, the polynomials $Q_i(x)$ are not all identical. This point is made clear with the aid of the following example.

The equation

$$L_3(y) \equiv y^{(3)} + ax^{-1}y^{(1)} + dx^{-3}y = 0, \qquad a \neq 0,\ d \neq 0, \tag{4.4}$$

is of the form (A′); its characteristic equation has a root, $\rho = 0$, of multiplicity three. There is a formal solution

$$y(x) = \sum_{\nu=0}^{\infty} y_\nu x^{-\nu}, \qquad y_0 = 1, \tag{4.4a}$$

where

$$y_{\lambda+1} = q(0)q(1)\cdots q(\lambda), \qquad q(\lambda) = \frac{d}{a(\lambda+1)} - \frac{\lambda(\lambda+2)}{a}.$$

The power series (4.4a) diverges so fast that exponential summability fails, and nothing of the type stated in our theorem will hold in connection with this formal solution.†

We shall outline very briefly the steps used in deriving the theorem referred to above. We apply to (A′) the transformation

$$y_j(x) = e^{Q(x)}x^r \sum_{h=0}^{j-1} \log^h x \left({}_h\eta_0 j^{-1} + \sum_{\nu=1}^{p} x^{(p-\nu)/p} {}_h\eta_\nu j^{-1}(x) \right). \tag{4.5}$$

This yields a differential system

$$\sum_{\beta=0}^{j-h-1} \sum_{k=0}^{n} \sum_{\iota=1}^{p} {}_\beta a_{\iota,k}^{h,\nu}(x)\, {}_{h+\beta}\eta_\iota^{j-1(k)}(x) = g^{h,\nu,j}(x), \tag{4.6}$$

$$h = 0, \cdots, j-1; \nu = 1, \cdots, p,$$

where the character of the coefficients is determined in detail ((T$_3$), p. 96). On writing $t = |t|e^{i\bar{t}}$, we suppose integrals

† In fact, it appears, as well, that no other method of summation will be applicable in this case.

$$\int_0^\infty, \qquad \int_0^t$$

to be extended over the ray $(0, \infty)$, of angle $\bar{t}$, denoted by $I(\bar{t})$, and the rectilinear segment $(0, t)$, respectively. If the variable x is so restricted that $\lim_t |e^{tx}t^\alpha| = 0$ (every $\alpha > 0$; $t \to \infty$ along $I(\bar{t})$), then, as is well known, we have formally

$$a(x) = \sum_{\nu=1}^{\infty} a_\nu x^{-\nu} = \int_0^\infty \tilde{a}(t)e^{tx}dt,$$

where

$$\tilde{a}(t) = \sum_{\nu=0}^{\infty} \tilde{a}_\nu t^{\nu-1}, \qquad \tilde{a}_\nu = \frac{(-1)^\nu a_\nu}{(\nu-1)!}.$$

On the basis of these facts Laplace integrals are introduced by means of the following transformations (refer to (4.5))

$$(4.7) \qquad {}_k\eta_\nu^{j-1}(x) = \int_0^\infty {}_k\tilde{\eta}_\nu^{j-1}(t)e^{tx}dt,$$

$$h = 0, \cdots, j-1; j = 1, \cdots, \phi; \nu = 1, \cdots, p.$$

This transformation, applied to (4.6), yields a system of integral equations

$$(4.8) \quad t^\phi {}_k\tilde{\eta}_\nu^{j-1}(t) = \sum_{\beta=0}^{j-h-1} \sum_{\xi=1}^{p} \int_0^t \beta^* c_\xi^{h,\nu}(t,\tau)_{h+\beta}\tilde{\eta}_\xi^{j-1}(\tau)d\tau + {}^*g^{h,\nu,j}(t),$$

$$h = 0, \cdots, j-1; \nu = 1, \cdots, p.$$

The character of the coefficients of this system is specified in (T_3), Lemma 3 (p. 103). In proving convergence of the formal solutions† ${}_k\tilde{\eta}_\xi^{j-1}(t)$, satisfying (4.8), the method of successive approximations (used with success by Horn in his more restricted problem) leads to apparently unsurmountable algebraic difficulties. The method pursued in (T_3) to overcome this difficulty was to establish a "dominant" system of integral equations; that is, a system from the convergence of whose formal solutions convergence of a set of solutions of (4.8) can be inferred. Such a system, as well as proof of convergence of the formal series solutions, is given in (T_3), pp. 103–113.

In order that the integrals (4.7) should lead to convergent factorial

† Power series in t.

series expansions,† in view of certain fundamental developments due to N. E. Nörlund,‡ it is necessary to prove that

$$(4.9) \qquad |{}_k\bar{\eta}_\nu^{j-1}(t)| < Ce^{q|t|}, \qquad k = 0, 1, \cdots, j-1; \nu = 1, \cdots, p,$$

along every ray $\bar{t}(=\angle t)$ in P_0 (C and q independent of $\bar{t}$ and $|t|$, positive and sufficiently great). Here P_0 is the complex t-plane, excluding a neighborhood of $t=0$ and excluding small sectors, each with vertex at $t=0$, and containing the zeros of $d_{0,0}t^{n-\phi}+d_{1,0}t^{n-\phi-1}\cdots+d_{n-\phi,0}=0$§ in their interiors. The inequalities (4.9) have been established in (T_3), pp. 114–116, following lines somewhat similar to those employed by Horn in his proof of analogous inequalities. The existence theorem stated in connection with (4.3) is inferred with the aid of (4.9).

5. **Some general remarks.** In connection with the equation (B), (§1) a remaining problem of considerable importance is that regarding the behaviour of solutions in the neighborhood of *isolated* points x_0 (on some real interval to which x is restricted) at which a number of roots $g_i(x)$ of the characteristic equation (3.2) coincide. We may have two intervals,

$$(u, x_0), \qquad (x_0, l), \qquad u < x_0 < l,$$

such that the two full sets of formal solutions, corresponding to these intervals, are essentially distinct from each other. Development of a comprehensive theory covering this situation is highly desirable.

Methods of the type of those introduced in (T_1) and (T_2) (in particular, the processes of "iterations") appear to be destined to be of utility in many similar problems.‖

In conclusion it would be appropriate to mention that the results established in (T_1) and (T_2) have found effective application in the field of non-linear differential equations.¶

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† And in fact to secure convergence of these integrals.

‡ N. E. Nörlund, *Leçons sur les Séries d'Interpolation*, Paris, 1926, pp. 206–208. Also see his theorem on p. 203.

§ Here $d_{n-\phi,0}\neq 0$. Moreover, by a transformation, $x^\sigma \exp Q(x)$ is supposed to be reduced to unity.

‖ In this connection should be mentioned some significant papers by C. Hurd (not yet published) in which such methods were applied to equations with several parameters.

¶ W. J. Trjitzinsky, *Theory of non-linear singular differential systems*, Transactions of this Society, vol. 42 (1937), pp. 225–321.

THE CONSTRUCTIVE SECOND NUMBER CLASS*

ALONZO CHURCH

The existence of at least a vague distinction between what I shall call the constructive and the non-constructive ordinals of the second number class, that is, between the ordinals which can in some sense be effectively built up to step by step from below and those for which this cannot be done (although there may be existence proofs), is, I believe, somewhat generally recognized. My purpose here is to propose an exact definition of this distinction and of the related distinction between constructive and non-constructive functions of ordinals in the second number class; where, again to speak vaguely, a function is constructive if there is a rule by which, whenever a value of the independent variable (or a set of values of the independent variables) is effectively given, the corresponding value of the dependent variable can be effectively obtained, effectiveness in the case of ordinals of the second number class being understood to refer to a step by step process of building up to the ordinal from below.

Much of the interest of the proposed definition lies, of course, in its absoluteness, and would be lost if it could be shown that it was in any essential sense relative to a particular scheme of notation or a particular formal system of logic. It is my present belief that the definition is absolute in this way—towards those who do not find this convincing the definition may perhaps be allowed to stand as a challenge, to find either a less inclusive definition which cannot be shown to exclude some ordinal which ought reasonably to be allowed as constructive, or a more inclusive definition which cannot be shown to include some ordinal of the second class which cannot be seen to be constructive.

It is believed that the distinction which it is proposed to develop between constructive and non-constructive ordinals (and functions of ordinals) should be of interest generally in connection with applications of the transfinite ordinals to mathematical problems. The relevance of the distinction is especially clear, however, in the case of applications of the ordinals to certain questions of symbolic logic (for example, various questions more or less closely related to the well known theorem of Gödel on undecidable propositions)†—this is

* An address delivered by invitation of the Program Committee at the Indianapolis meeting of the Society, December 29, 1937.

† Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und*

because of the criterion of effectiveness or "definiteness" which necessarily applies to the rules of procedure of a formal system of logic.* The distinction is also presumably relevant to proposals to classify recursive functions of natural numbers according to ordinals of the second number class†—because the possibility of an effective step by step calculation of the values of the function may reasonably be taken as an essential part of the notion of a recursive function.

For present purposes it will be convenient to make a minor departure from the established terminology, using "second number class" in such a sense that the first number class is included as a part of the second number class (thus avoiding the sometimes awkward phrase, "first and second number classes"). On this basis, the *second number class* may be described as the simply ordered set which results when we take 0 as the first (or least) element of the set and allow the two following processes of generation: (1) given any element of the set, to generate the element which next follows it (the least element greater than it); (2) given any infinite increasing sequence of elements, of the order type of the natural numbers, to generate the element which next follows the sequence (the least element greater than every element of the sequence).‡ The elements of the set are *ordinals*. The ordinal which next follows a given ordinal is the *successor* of that ordinal. The ordinal which next follows a given infinite increasing sequence of ordinals, of the order type of the natural numbers, is the *upper limit* of that sequence. The ordinal which has a given ordinal as successor is the *predecessor* of that ordinal. An infinite increasing sequence of ordinals, of the order type of the natural numbers, which has a given ordinal as upper limit is a *fundamental sequence* of that

verwandter Systeme I, Monatshefte für Mathematik und Physik, vol. 38 (1931), pp. 173–198. See also *On Undecidable Propositions of Formal Mathematical Systems*, mimeographed lecture notes, Princeton, N. J., 1934.

* There is some current tendency to apply the name "logic" to schemes which are similar to accepted systems of symbolic logic but involve one or more rules of procedure which lack this characteristic of effectiveness. Such schemes may perhaps be of interest as abstract definitions of classes of formulas, but they cannot in my opinion be called "logics" except by a drastic (and possibly misleading) change in the usual meaning of that word. For they do not provide an *applicable* criterion as to what constitutes a valid proof.

† Cf. David Hilbert, *Über das Unendliche*, Mathematische Annalen, vol. 95 (1926), pp. 161–190; Wilhelm Ackermann, *Zum Hilbertschen Aufbau der reellen Zahlen*, Mathematische Annalen, vol. 99 (1928), pp. 118–133.

‡ This definition of the second number class is selected as fundamental because it represents the way in which, ordinarily, the ordinals actually arise in applications to mathematical problems—in particular the way in which they arise in connection with those questions of symbolic logic to which reference was just made.

ordinal. Every ordinal except 0 has either a predecessor or a fundamental sequence but not both; in the first case the ordinal is of the *first kind*, in the second case of the *second kind*.

As a definition of the distinction between constructive and non-constructive ordinals in the second number class might be proposed the following:

An ordinal ξ is constructive if it is possible to devise a system of notation which assigns a unique notation to every ordinal less than or equal to ξ and, associated with the system of notation, three effective processes by which, respectively, (1) given the notation for any ordinal it can be determined whether the ordinal is of the first or the second kind, (2) given the notation for any ordinal of the first kind the notation for the predecessor of the ordinal can be obtained, (3) given the notation for any ordinal of the second kind a fundamental sequence of that ordinal can be obtained, in the sense of an effective process for calculating the notations for the successive terms of the sequence.*

It will be seen that the contemplated system of notation for ordinals less than or equal to ξ will then also admit an effective process by which, given the notations for any two ordinals, it can be determined which ordinal is greater.

Moreover, of course this contemplated system of notation is required (by definition) to yield an effective simultaneous selection, for every ordinal μ of the second kind less than or equal to ξ, of one out of the various fundamental sequences of μ. In fact, such an assignment of a unique fundamental sequence to every ordinal of the second kind less than or equal to ξ is held evidently to be a necessary consequence of the step by step process of building up to ξ from below which was first taken as characterizing constructiveness.

The present definition of constructiveness of ξ (that there exists a system of notation, of the kind described, for the ordinals less than or equal to ξ) is thought to correspond satisfactorily to the vaguer notion with which we began, and also to be satisfactorily exact, except for one thing, the vagueness of the notion of an effective process.

This notion of an effective process occurs frequently in connection with mathematical problems, where it is apparently felt to have a clear meaning, but this meaning is commonly taken for granted

* A similar but essentially different property of an ordinal ξ is employed by S. C. Kleene; see *On notation for ordinal numbers*, abstract in this Bulletin, vol. 43 (1937), p. 41.

without explanation. For our present purpose it is desirable to give an explicit definition.

Perhaps the most convincing form in which this definition of an effective process can be put results from the adoption of an idea of Turing.* A process is effective if it is possible to devise a calculating machine, with a finite number of parts of finite size, which, with the aid of an endless tape, running through the machine, on which symbols are printed, will carry out the process in any particular case—of course only a finite portion of the tape being used in any particular case. (It will be seen that a human calculator, provided with pencil and paper and explicit instructions, can be thought of as a machine of this kind, the paper taking the place of the tape.)

In the case of a process which, applied to a natural number, yields a natural number, another form of the definition of effectiveness is that the process is effective if it corresponds to an arithmetic function which is recursive in the most general sense.† This definition can be extended to processes upon the formulas of an arbitrary system of notation by employing the now familiar device of representing the formulas by Gödel numbers.

Still another form of the definition of an effective process is obtained by replacing the condition of general recursiveness by the condition of λ-definability.‡

The equivalence of these three definitions is established in papers by Kleene and Turing.§

This completes the explanation of the proposed definition of the distinction between constructive and non-constructive ordinals in the second number class. In order, however, to obtain a definition of the related distinction between constructive and non-constructive functions of ordinals, and in order to establish that not every ordinal in the second number class is constructive, it is desirable to extend the notion of λ-definition, which was first introduced for positive in-

* A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, (2), vol. 42 (1936–1937), pp. 230–265.

† I.e., general recursive in the sense of Herbrand and Gödel. Cf. Kurt Gödel, *On Undecidable Propositions of Formal Mathematical Systems*, pp. 26–27; S. C. Kleene, *General recursive functions of natural numbers*, Mathematische Annalen, vol. 112 (1935–1936), pp. 727–742.

‡ Cf. Alonzo Church, *An unsolvable problem of elementary number theory*, American Journal of Mathematics, vol. 58 (1936), pp. 345–363.

§ S. C. Kleene, *λ-definability and recursiveness*, Duke Mathematical Journal, vol. 2 (1936), pp. 340–353; A. M. Turing, *Computability and λ-definability*, The Journal of Symbolic Logic, vol. 2 (1937), pp. 153–163.

tegers, to the transfinite ordinals by introducing appropriate formulas (of the λ-formalism) to represent the ordinals. For the constructive ordinals of the second number class this is done as follows.

Using an arrow to mean "stands for" or "is an abbreviation for," let

$$0_o \rightarrow \lambda m.m(1),$$
$$S_o \rightarrow \lambda am.m(2, a),$$
$$L \rightarrow \lambda arm.m(3, a, r),$$

where,

$$1 \rightarrow \lambda fx.f(x),$$
$$2 \rightarrow \lambda fx.f(f(x)),$$
$$3 \rightarrow \lambda fx.f(f(f(x))).$$

Then let a class of formulas, to be called *ordinal-formulas*, and a relation $<$ between ordinal-formulas be defined simultaneously by induction as follows (1–6):

1. If $\boldsymbol{a}$ is an ordinal-formula and $\boldsymbol{b}$ conv $\boldsymbol{a}$, then $\boldsymbol{b}$ is an ordinal-formula; further any ordinal-formula which bears the relation $<$ to $\boldsymbol{a}$ bears that relation also to $\boldsymbol{b}$; further $\boldsymbol{b}$ bears the relation $<$ to any ordinal-formula to which $\boldsymbol{a}$ bears that relation.

2. 0_o is an ordinal-formula.

3. If $\boldsymbol{a}$ is an ordinal-formula, $S_o(\boldsymbol{a})$ is an ordinal-formula, and $\boldsymbol{a} < S_o(\boldsymbol{a})$.

4. If $\boldsymbol{r}$ is a well-formed formula and each of the formulas in the infinite list, $\boldsymbol{r}(0_o)$, $\boldsymbol{r}(S_o(0_o))$, $\boldsymbol{r}(S_o(S_o(0_o)))$, $\cdots$, is an ordinal-formula and bears the relation $<$ to the formula which follows it in the list, then $L(0_o, \boldsymbol{r})$ is an ordinal-formula, and each of the formulas in the infinite list bears the relation $<$ to $L(0_o, \boldsymbol{r})$.

5. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are ordinal-formulas and $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{b} < \boldsymbol{c}$, then $\boldsymbol{a} < \boldsymbol{c}$.

6. The ordinal-formulas are the smallest class of formulas possible consistently with 1–5, and the relation $<$ subsists between two ordinal formulas only when compelled to do so by 1–5.

It will be seen that under the relation $<$ the ordinal-formulas form, not a simply ordered, but a partially ordered set.

Then an assignment of formulas to represent ordinals in the second number class is defined by induction as follows (i–v):

i. If $\boldsymbol{a}$ represents an ordinal a, and $\boldsymbol{b}$ conv $\boldsymbol{a}$, then $\boldsymbol{b}$ also represents a.

ii. 0_o represents the ordinal 0.

iii. If a represents an ordinal a, then $S_o(a)$ represents the successor of a.

iv. If r is a well-formed formula, and each of the formulas in the infinite list, $r(0_o)$, $r(S_o(0_o))$, $r(S_o(S_o(0_o)))$, $\cdots$, is an ordinal-formula and bears the relation $<$ to the formula which follows it in the list, and if the formulas in the infinite list represent respectively the ordinals b_0, b_1, b_2, $\cdots$, then $L(0_o, r)$ represents the upper limit of the sequence b_0, b_1, b_2, $\cdots$.*

v. A formula represents an ordinal only if compelled to do so by i–iv.

Evidently, every formula which represents an ordinal is an ordinal-formula (as previously defined) and every ordinal-formula represents an ordinal. If the relation $<$ (between ordinal-formulas) holds between two given ordinal-formulas, the relation *less than* (between ordinals) must hold between the ordinals which they represent—but not conversely.

Moreover, under this assignment of formulas to represent ordinals in the second number class, every formula which represents an ordinal has a normal form, and no formula represents more than one ordinal, as has been proved by Church and Kleene.† In general, however, the same ordinal is represented by an infinite number of non-interconvertible formulas.

Let us call an ordinal in the second number class λ-*definable* if there is at least one formula which represents it under the foregoing scheme. Then the class of λ-definable ordinals coincides with the class of constructive ordinals previously defined.

For the ordinal-formulas in principal normal form‡ which bear the relation $<$ to a given ordinal-formula form a simply ordered set un-

* This constitutes a minor but essential correction to the assignment of formulas to represent ordinals which is proposed by Alonzo Church and S. C. Kleene, *Formal definitions in the theory of ordinal numbers*, Fundamenta Mathematicae, vol. 28 (1937), pp. 11–21. The correction is minor in the sense that no further changes in the cited paper are necessitated by it.

The correction is regarded as essential from the point of view of intuitive justification. Since this address was delivered, however, I have seen a proof by Kleene that the two definitions of λ-definability of ordinals in the second number class (the definition given here and our previous definition) are equivalent.

† In the paper cited in the preceding footnote.

‡ For definition of the terms *normal form* and *principal normal form* reference may be made to *An unsolvable problem of elementary number theory*. The normal form of a formula is ambiguous only to the extent of possible alphabetical changes of the bound variables which appear, but it is sometimes convenient to remove even this ambiguity by adopting a device due to Kleene by which a particular one of the normal forms of a formula is designated as the principal normal form.

der the relation $<$, as can be proved by transfinite induction. If ξ is a λ-definable ordinal, and $\mathbf{\xi}$ is a formula in principal normal form which represents ξ, then $\mathbf{\xi}$ together with the ordinal-formulas in principal normal form which bear the relation $<$ to $\mathbf{\xi}$ constitute a system of notation for the ordinals less than or equal to ξ which is of the kind required by the definition of constructiveness. Hence every λ-definable ordinal is constructive.

Moreover, if ξ is a constructive ordinal, the system of notation for ordinals less than or equal to ξ which is referred to in the definition of constructiveness can be replaced by an assignment of a positive integer to each ordinal less than or equal to ξ, the notation for each ordinal being replaced by the Gödel number of the notation. And then the positive integer assigned to each ordinal may in turn be replaced by the formula (of the λ-formalism) which represents that positive integer.* Thus there is correlated to each ordinal less than or equal to ξ a formula (of the λ-formalism), and in fact one of the formulas which represent the positive integers. In view of the definition of the constructiveness of ξ, and since every effective function of positive integers is λ-definable, there will be three formulas $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{f}$, having the following properties: if $\boldsymbol{m}$ is any formula (of the λ-formalism) which is correlated (as just described) to an ordinal μ less than or equal to ξ, then $\boldsymbol{K}(\boldsymbol{m})$ conv 1 if μ is of the first kind, $\boldsymbol{K}(\boldsymbol{m})$ conv 2 if μ is of the second kind, $\boldsymbol{K}(\boldsymbol{m})$ conv 3 if μ is 0_o; also if μ is of the first kind, $\boldsymbol{P}(\boldsymbol{m})$ conv the formula correlated to the predecessor of μ; and if μ is of the second kind and $\boldsymbol{n}$ is a formula which represents a finite ordinal n,† then $\boldsymbol{f}(\boldsymbol{m}, \boldsymbol{n})$ conv the formula correlated to the $(1+n)$th term of that fundamental sequence of μ which is given by the effective process (3) referred to in the definition of the constructiveness of ξ. Using a theorem of Kleene,‡ one may obtain a formula $\boldsymbol{T}$ having the following conversion properties, where $\boldsymbol{m}$ is any formula which represents a positive integer:

$$\boldsymbol{T}(\boldsymbol{m}) \text{ conv } S_o(\boldsymbol{T}(\boldsymbol{P}(\boldsymbol{m}))), \text{ if } \boldsymbol{K}(\boldsymbol{m}) \text{ conv } 1;$$

$$\boldsymbol{T}(\boldsymbol{m}) \text{ conv } L(0_o, \lambda n \boldsymbol{T}(\boldsymbol{f}(\boldsymbol{m}, n))), \text{ if } \boldsymbol{K}(\boldsymbol{m}) \text{ conv } 2;$$

$$\boldsymbol{T}(\boldsymbol{m}) \text{ conv } 0_o, \text{ if } \boldsymbol{K}(\boldsymbol{m}) \text{ conv } 3.$$

* For the representation of positive integers by formulas (of the λ-formalism) see, for example, *An unsolvable problem of elementary number theory*.

† The finite ordinals are here taken as distinct from the corresponding non-negative integers, as is done in the cited paper of Church and Kleene. In view of the formula $\mathfrak{Z}$ derived in that paper (page 19), there is no difficulty caused here by taking n to be a finite ordinal rather than a positive integer.

‡ *λ-definability and recursiveness*, Theorem 19 and footnote 17.

Then if $\boldsymbol{m}$ is the formula which is correlated to an ordinal μ less than or equal to ξ, the formula $\boldsymbol{T}(\boldsymbol{m})$ will represent the ordinal μ (proof by transfinite induction). In particular if $\boldsymbol{x}$ is the formula which is correlated to ξ, then $\boldsymbol{T}(\boldsymbol{x})$ will represent ξ. Hence every constructive ordinal is λ-definable.

Now let a function $F(x_1, x_2, \cdots, x_r)$ of ordinals in the second number class be called λ-*definable* if there is a formula $\boldsymbol{F}$ such that whenever the formulas $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r$ represent the ordinals $x_1, x_2, \cdots, x_r$ respectively, the formula $\boldsymbol{F}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r)$ will represent the ordinal $F(x_1, x_2, \cdots, x_r)$. As a definition of the notion of a constructive function of ordinals in the second number class, it is proposed simply to identify this notion with that of a λ-definable function of ordinals in the second number class. This is rendered plausible by the known properties of the λ-formalism, and no definition with a more direct appeal suggests itself.

It has been proved by Church and Kleene* that a large class of functions of ordinals are λ-definable, including addition, multiplication, exponentiation, the function ϵ_x, a predecessor function of ordinals, and others. The function $\phi(x, y)$ of ordinals in the second number class whose value is the ordinal 0 when x is less than y, and 1 when x is equal to y, and 2 when x is greater than y is, however, demonstrably not λ-definable.† This is taken to mean that from a strictly finitary point of view the second number class as a simply ordered set is inadmissible and must be replaced by a partially ordered set which has the structure of the set of ordinal-formulas under the relation $<$. Many of the classical theorems about the second number class may, however, be represented by corresponding finitary theorems about this partially ordered set.

In order to establish the existence of non-constructive ordinals in the second number class it is sufficient to observe that the class of all formulas (of the λ-formalism) is enumerable, and hence (by a non-constructive argument) that the class of all λ-definable ordinals is enumerable, whereas the class of all ordinals in the second number class is non-enumerable.

Let ω_1 be the least non-constructive ordinal in the second number

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* Loc. cit.

† This is not surprising. It is, for instance, not difficult to give examples of pairs of constructive definitions of ordinals such that the question whether the ordinals defined are equal, or which of the two is greater, depends on this or that unsolved problem of number theory; and indeed this may be done without employing any ordinal greater than ω^2.

class. Then evidently every ordinal greater than ω_1 in the second number class is also non-constructive.

It is readily proved that ω_1 cannot be of the first kind. Thus by an indirect argument the existence of a fundamental sequence of ω_1 is established. But demonstrably there cannot be an effective process of calculating the successive terms of a fundamental sequence of ω_1.*

From a finitary point of view, ω_1 belongs to the third number class but not the second.

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* If it be required that the process of calculating the successive terms of the sequence shall not only provide a step by step process of building up to each term from below, but shall also effectively exhibit the increasing character of the sequence by including the step by step process of building up to any term as a part of the corresponding process for each subsequent term, the impossibility is an immediate consequence of the definition of ω_1. If, however, the condition be omitted that the process effectively exhibit the increasing character of the sequence in this way, the proof of impossibility depends on the theorem of Kleene previously referred to concerning the equivalence of the two definitions of λ-definability of ordinals in the second number class.

A REMARK ON REPRESENTATIONS OF GROUPS*

TADASI NAKAYAMA

The purpose of this short note is to remark that we can state an analog of a famous theorem of Frobenius† on the induced characters of a finite group also for the representations of a general group.‡ This extension has not yet been explicitly stated, so far as I know, although it can be quite easily verified.

Let $\mathfrak{g}$ be a group, and let $\mathfrak{h}$ be a subgroup (*of a finite or infinite index*) of $\mathfrak{g}$.

DEFINITION. Let $F(x)$ and $f(\xi)$ be almost periodic (a. p.) functions (with complex numbers as values) on $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then we define the compositions of $F(x)$ and $f(\xi)$ by

$$f \times F(x) = M_{\xi\mathfrak{h}}[f(\xi)F(\xi^{-1}x)],$$
$$F \times f(x) = M_{\xi\mathfrak{h}}[F(x\xi^{-1})f(\xi)];$$

where $M_{\xi\mathfrak{h}}$ means the construction of the mean with respect to a variable ξ in $\mathfrak{h}$. *Here $f\times F(x)$ and $F\times f(x)$ are a. p. functions on $\mathfrak{g}$, and they are linear with respect to both factors, $f(\xi)$ and $F(x)$.*

If $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$ are three subgroups of $\mathfrak{g}$ such that $\mathfrak{h}_i \subseteqq \mathfrak{h}_k$ or $\mathfrak{h}_i \supseteqq \mathfrak{h}_k$ for every i, $k=1, 2, 3$, then

$$(f_1 \times f_2) \times f_3 = f_1 \times (f_2 \times f_3)$$

for a. p. functions $f_1(\xi_1)$, $f_2(\xi_2)$, $f_3(\xi_3)$ on $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$ respectively.

(Both sides of the equality are a. p. functions on the greatest among the $\mathfrak{h}_i$.) This product we denote by $f_1\times f_2\times f_3$.

All these statements we can prove by a procedure similar to that

* Presented to the Society, October 30, 1937.

† G. Frobenius, *Ueber Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen*, Berlin Sitzungsberichte, 1898; H. Weyl, *Gruppentheorie und Quantenmechanik*; J. Levitzki, *Ueber vollständig reduzible Ringe und Unterringe*, Mathematische Zeitschrift, vol. 33 (1931).

‡ J. von Neumann, *Almost periodic functions in a group*, Transactions of this Society, vol. 36 (1934). Cf. also S. Bochner and J. von Neumann, *Almost periodic functions in groups*, II, ibid., vol. 37 (1935); W. Maak, *Eine neue Definition der fastperiodischen Funktionen*, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 11 (1936); B. L. van der Waerden, *Gruppen von linearen Transformationen*, Ergebnisse der Mathematik, vol. 4 (1935).

By a representation of a group we understand always a bounded one in the field of complex numbers.

of von Neumann in the paper cited. Therefore we can consider the ring $\mathfrak{R}_\mathfrak{h}$ of a. p. functions on $\mathfrak{h}$ as a (right and left) operator-ring of the ring $\mathfrak{R}_\mathfrak{g}$ of a. p. functions on $\mathfrak{g}$. If $\mathfrak{M}$ and $\mathfrak{m}$ are submoduli of $\mathfrak{R}_\mathfrak{g}$ and $\mathfrak{R}_\mathfrak{h}$ respectively, then we denote by $\mathfrak{M}\times\mathfrak{m}$ the submodule of $\mathfrak{R}_\mathfrak{g}$ generated by the elements $(F\times f(x),\ F(x)\in\mathfrak{M},\ f(\xi)\in\mathfrak{m})$. We define $\mathfrak{m}\times\mathfrak{M}$ in a similar manner.

DEFINITION. Let $\mathfrak{n}$ be a left ideal of $\mathfrak{R}_\mathfrak{h}$ with a finite rank with respect to the field Ω of complex numbers.* Then $\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}$ is obviously a left ideal of $\mathfrak{R}_\mathfrak{g}$ (with a finite or infinite rank with respect to Ω). We call $\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}$ *the left ideal of* $\mathfrak{R}_\mathfrak{g}$ *induced by* $\mathfrak{n}$.

As is well known, there is an idempotent element $c(\xi)$ in $\mathfrak{R}_\mathfrak{h}$ such that $\mathfrak{n}=\mathfrak{R}_\mathfrak{h}\times c$; it is $f\times c=f$ for every $f(\xi)$ in $\mathfrak{n}$. If $F(x)\in\mathfrak{R}_\mathfrak{g}$ and $f(\xi)\in\mathfrak{n}$, then we have $F\times f=F\times(f\times c)=(F\times f)\times c\in\mathfrak{R}_\mathfrak{g}\times c$. Therefore $\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}\subseteq\mathfrak{R}_\mathfrak{g}\times c$, and this implies $\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}=\mathfrak{R}_\mathfrak{g}\times c$.

The ideal $\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}$ *consists of all functions* $G(x)\in\mathfrak{R}_\mathfrak{g}$ *such that for every* $x\in\mathfrak{g}$ *the function* $G(x\xi)$ *of* $\xi\in\mathfrak{h}$ *lies in* $\mathfrak{n}$.

Let $G(x)$ have the property stated above. Then

$$M_{\eta\in\mathfrak{h}}[G(x\xi\eta^{-1})c(\eta)]=G(x\xi);$$

in particular,

$$G\times c(x)=M_{\eta\in\mathfrak{h}}[G(x\eta^{-1})c(\eta)]=G(x),$$

that is, $G(x)\in\mathfrak{R}_\mathfrak{g}\times c=\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}$.

The other half of the statement is obvious.

Now we have the following theorem:

THEOREM. *Let* $\mathfrak{n}$ *be a minimal left ideal of* $\mathfrak{R}_\mathfrak{h}$, *and* $\mathfrak{d}$ *the irreducible representation of* $\mathfrak{h}$ *defined by* $\mathfrak{n}$. *Let* $\mathfrak{D}$ *be an irreducible representation of* $\mathfrak{g}$, *and* $\mathfrak{S}$ *the two-sided ideal of* $\mathfrak{R}_\mathfrak{g}$ *belonging to* $\mathfrak{D}$.† *We denote by* $\mathfrak{D}(\mathfrak{h})$ *a representation of* $\mathfrak{h}$ *formed by the matrices in* $\mathfrak{D}$ *which correspond to the elements of* $\mathfrak{h}$. *If the number of the irreducible constituents of* $\mathfrak{D}(\mathfrak{h})$ *equivalent to* $\mathfrak{d}$ *is* g, *then the representation of* $\mathfrak{g}$ *defined by the* $\mathfrak{S}$*-component* $\mathfrak{S}\times\mathfrak{N}=\mathfrak{S}\times\mathfrak{n}$ *of the induced left ideal* $\mathfrak{N}=\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}$ *consists of just* g *irreducible constituents* (*equivalent to* $\mathfrak{D}$).

PROOF.‡ Let $E(x)$ denote the principal unit of $\mathfrak{S}$, and put

* Note that a submodule of $\mathfrak{R}_\mathfrak{h}$ ($\mathfrak{R}_\mathfrak{g}$) with a finite rank with respect to Ω is a left ideal if and only if it is an $\mathfrak{h}$-($\mathfrak{g}$-)left-module; where we define the multiplication of $\alpha\in\mathfrak{h}$ and $f(\xi)\in\mathfrak{R}_\mathfrak{h}$, ($\alpha\in_\mathfrak{g}$ and $F(x)\in\mathfrak{R}_\mathfrak{g}$), by $\alpha\cdot f(\xi)=f(\alpha^{-1}\xi)$, $(a\cdot F(x)=F(a^{-1}x))$. Then we have $(\alpha\cdot f)\times F=\alpha\cdot(f\times F)$, $(a\cdot F)\times f=a\cdot(F\times f)$, and so on.

† Linear aggregates of the matric elements of $\mathfrak{D}$ form a two sided ideal of $\mathfrak{R}_\mathfrak{g}$, which is isomorphic to a matric ring.

‡ The following proof is only a slight modification of the proof in Weyl, loc. cit.

$\mathfrak{N}=\mathfrak{S}\times\mathfrak{N}=E\times\mathfrak{N}(=\mathfrak{N}\times\mathfrak{S}=\mathfrak{N}\times E)$. In the case $\mathfrak{N}=0$ the theorem is obvious. We assume therefore $\mathfrak{N}\neq 0$, that is, $c\times E\neq 0$.

If we denote the degree of the representation $\mathfrak{d}$ by r, then the two-sided ideal $\mathfrak{s}$ of $\mathfrak{R}_\mathfrak{h}$ belonging to $\mathfrak{d}$ is a direct sum of r minimal left ideals operator-isomorphic to $\mathfrak{n}$:

$$\mathfrak{s}=\mathfrak{n}^{(1)}+\mathfrak{n}^{(2)}+\cdots+\mathfrak{n}^{(r)},\qquad \mathfrak{n}^{(i)}\cong\mathfrak{n}.$$

Put $\mathfrak{N}^{(i)}=\mathfrak{R}_\mathfrak{g}\times\mathfrak{n}^{(i)}$ and $\tilde{\mathfrak{N}}^{(i)}=\mathfrak{S}\times\mathfrak{N}^{(i)}=\mathfrak{S}\times\mathfrak{n}^{(i)}$. It is easy to see that $\mathfrak{R}_\mathfrak{g}\times\mathfrak{s}$ is the direct sum $\mathfrak{N}^{(1)}+\mathfrak{N}^{(2)}+\cdots+\mathfrak{N}^{(r)}$, and also that $\mathfrak{S}\times\mathfrak{R}_\mathfrak{g}\times\mathfrak{s}=\mathfrak{S}\times\mathfrak{s}=\tilde{\mathfrak{N}}^{(1)}+\tilde{\mathfrak{N}}^{(2)}+\cdots+\tilde{\mathfrak{N}}^{(r)}$. Now suppose that $\tilde{\mathfrak{N}}$ is a direct sum of h minimal left ideals. (Our purpose is to show $h=g$.) Then each of $\tilde{\mathfrak{N}}^{(i)}$ has the same property: $\tilde{\mathfrak{N}}^{(i)}=\mathfrak{L}_1^{(i)}+\mathfrak{L}_2^{(i)}+\cdots+\mathfrak{L}_h^{(i)}$, for it is operator-isomorphic to $\tilde{\mathfrak{N}}$.

Let $e(\xi)$ be the principal unit of $\mathfrak{s}$. We have $\mathfrak{S}\times\mathfrak{s}=\mathfrak{R}_\mathfrak{g}\times(e\times E)$ and

$$e\times E=E\times e\times E=E\times e,\qquad (e\times E)^2=E\times e\times E=e\times E.$$

Moreover

$$\mathfrak{S}=\mathfrak{S}\times\mathfrak{s}+\mathfrak{L}^*=\mathfrak{L}_1^{(1)}+\mathfrak{L}_2^{(1)}+\cdots+\mathfrak{L}_h^{(r)}+\mathfrak{L}^*$$

for a suitably chosen left ideal $\mathfrak{L}^*$ of $\mathfrak{S}$. From this decomposition we see in the usual manner that $(e\times E)\times\mathfrak{R}_\mathfrak{g}\times(e\times E)$ is a matric ring of degree rh (over Ω):

$$(e\times E)\times\mathfrak{R}_\mathfrak{g}\times(e\times E)=\sum_{\substack{k,i=1,\cdots,r\\ \mu,\lambda=1,\cdots,h}} C_{\mu\lambda}^{(k)(i)}\Omega,$$

$$(e\times E)\times\mathfrak{L}_\lambda^{(i)}=\sum_{k,\mu} C_{\mu\lambda}^{(k)(i)}\Omega,$$

($C_{\mu\lambda}^{(k)(i)}(x)$ being matric units). Here $(e\times E)\times\mathfrak{L}_\lambda^{(i)}=e\times(E\times\mathfrak{L}_\lambda^{(i)})=e\times\mathfrak{L}_\lambda^{(i)}$, and therefore $(e\times\mathfrak{L}_\lambda^{(i)}:\Omega)=rh$.

On the other hand $\mathfrak{L}_\lambda^{(i)}$ is, considered as an $\mathfrak{h}$-left-module, completely reducible, for it defines a representation of $\mathfrak{h}$ equivalent to $\mathfrak{D}(\mathfrak{h})$. Let $\mathfrak{L}_\lambda^{(i)}=\mathfrak{M}_1+\mathfrak{M}_2+\cdots+\mathfrak{M}_l$ be its decomposition into simple (minimal) submoduli. According to our assumption just g of $\mathfrak{M}_j$ are operator-isomorphic to $\mathfrak{n}$ with respect to $\mathfrak{h}$, and therefore also with respect to $\mathfrak{R}_\mathfrak{h}$.† This implies $(e\times\mathfrak{L}_\lambda^{(i)}:\Omega)=rg$.

Comparing with the above result, we obtain $h=g$.

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† A submodule, with a finite rank, of $\mathfrak{R}_\mathfrak{g}$ is an $\mathfrak{R}_\mathfrak{h}$-left-module if and only if it is an $\mathfrak{h}$-left-module. Two such moduli are operator-isomorphic with respect to $\mathfrak{R}_\mathfrak{h}$ if and only if they are so with respect to $\mathfrak{h}$. The same holds for the isomorphism between such a module and a left ideal of $\mathfrak{R}_\mathfrak{h}$.

SOME PROPERTIES OF FUNCTIONS OF EXPONENTIAL TYPE

RICHARD DUFFIN AND A. C. SCHAEFFER

Suppose that $f(z)$ is an entire function such that

$$f(z) = O(e^{\lambda|z|}),$$

and on the real axis $f(z)$ is real and bounded by 1. First it is shown that the function $\cos \lambda z - f(z)$ cannot have complex zeros. Moreover its real zeros are simple at the points where the strict inequality $|f(z)| < 1$ is satisfied. This theorem is then used to find a "best possible" dominant over the complex plane of the class of functions $f(z)$. Finally it is shown that these results contain two theorems of S. Bernstein.

THEOREM 1. *Let $f(z)$ be an entire function of $z = x + iy$, real for real z, and satisfying the conditions*:

$$|f(x)| \leqq 1 \tag{1}$$

on the real axis, and

$$|f(z)| = O(e^{\lambda|z|}), \qquad \lambda > 0, \tag{2}$$

uniformly over the entire plane. Then for every real α the function

$$\cos(\lambda z + \alpha) - f(z)$$

has only real zeros, or vanishes identically. Moreover all the zeros are simple, except perhaps at points on the real axis where $f(x) = \pm 1$.

In the proof of Theorem 1 we shall use the following result of Pólya and Szegö,* which we state as a lemma.

LEMMA 1. *If $f(z)$ satisfies the conditions of Theorem* 1, *then actually*

$$|f(z)| \leqq e^{\lambda|y|}. \tag{3}$$

PROOF. By the hypotheses of Theorem 1 the function $f(z)e^{i\lambda z}$ is bounded on the positive halves of the real and imaginary axes and is $O(e^{\lambda|z|})$ in the angular region between them. Then by the Phragmén-Lindelöf† principle $f(z)e^{i\lambda z}$ is bounded throughout the first quadrant. In the same way one shows that it is bounded in the second quadrant.

* G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 36, prove this by a different method.

† E. C. Titchmarsh, *Theory of Functions*, p. 177.

Then $f(z)e^{\Omega z}$ is bounded by 1 on the real axis and by some constant in the upper half-plane. Again applying the Phragmén-Lindelöf principle, this time with the angular region the upper half-plane, we see that $|f(z)e^{\Omega z}| \leqq 1$ in the upper half-plane, so (3) is true for $y \geqq 0$. When we use the function $f(z)e^{-\Omega z}$ the same method shows that (3) is true for $y \leqq 0$.

To prove Theorem 1 it will be sufficient to consider the function $\cos \lambda z - f(z)$, that is, the case $\alpha = 0$. Let ϵ be some positive number less than 1, and consider the function

$$f_\epsilon(z) = \frac{\sin \lambda \epsilon z}{\lambda \epsilon z}(1-\epsilon) f((1-\epsilon)z).$$

By Lemma 1,

$$|f_\epsilon(z)| \leqq \frac{e^{\lambda\epsilon|y|}}{\lambda\epsilon|z|}(1-\epsilon)e^{\lambda|y|(1-\epsilon)} < \frac{e^{\lambda|y|}}{\lambda\epsilon|z|}; \tag{4}$$

so if y_0 is sufficiently large, we have, on the lines $y = \pm y_0$,

$$|f_\epsilon(z)| < |\cos \lambda z|.$$

It follows from (4) that, if K is a sufficiently large positive integer,

$$|f_\epsilon(z)| < |\cos \lambda z|$$

on the lines $x = \pm K\pi/\lambda$.

Let ζ be a closed rectangular contour consisting of segments of the lines $x = \pm K\pi/\lambda$, $y = \pm y_0$. We have shown that $|\cos \lambda z| > |f_\epsilon(z)|$ on ζ, so by Rouché's theorem* the function

$$\cos \lambda z - f_\epsilon(z)$$

has the same number of zeros in ζ as $\cos \lambda z$, that is, $2K$ zeros. On the real axis $|f_\epsilon(x)| \leqq 1-\epsilon$, so at the points $\nu\pi/\lambda$, $(\nu = 0, \pm 1, \pm 2, \cdots)$, we have $|f_\epsilon(z)| < |\cos \lambda z|$. Thus $\cos \lambda z - f_\epsilon(z)$ is alternately plus and minus at the $2K+1$ points $\nu\pi/\lambda$, $(\nu = -K, -K+1, \cdots, K)$; so inside ζ it has at least $2K$ real zeros. But we have shown that there are exactly $2K$ zeros of $\cos \lambda z - f_\epsilon(z)$ in ζ. Hence there are no complex zeros, and there is exactly one (simple) zero in each interval $(\nu\pi/\lambda, (\nu+1)\pi/\lambda)$, $(\nu = -K, \cdots, K-1)$. Taking larger values of y_0 and K we see that $\cos \lambda z - f_\epsilon(z)$ has exclusively real and simple zeros, which lie in the intervals $\nu\pi/\lambda < z < (\nu+1)\pi/\lambda$, ($\nu$ integer, $-\infty < \nu < \infty$).

When $\epsilon \to 0$ the function $\cos \lambda z - f_\epsilon(z)$ approaches $\cos \lambda z - f(z)$

* E. C. Titchmarsh, loc. cit., p. 116.

uniformly in every bounded domain. But if the latter function is not identically zero it follows from a theorem of Hurwitz* that its zeros are limit points of the zeros of $\cos \lambda z - f_\epsilon(z)$. Thus $\cos \lambda z - f(z)$ cannot have non-real zeros; moreover it has an infinite number of real zeros which are all simple, except those at the points $\nu\pi/\lambda$ if $f(\nu\pi/\lambda) = (-1)^\nu$. Every interval $\nu\pi/\lambda < z < (\nu+1)\pi/\lambda$ at the endpoints of which $|f(z)| < 1$ contains exactly one zero. If $f(\nu\pi/\lambda) = (-1)^\nu$, we have a double zero at $\nu\pi/\lambda$ but no further zeros in the interior or at the endpoints of the interval $((\nu-1)\pi/\lambda, (\nu+1)\pi/\lambda)$. This proves Theorem 1.

We have shown that, if $f(z)$ satisfies the conditions of Theorem 1, then actually the inequality (3) is satisfied. There is, however, no such function for which (3) becomes an equality at points off the real axis. Using Theorem 1 we can show that the stronger inequality

$$|f(z)| \leqq \cosh \lambda y, \tag{5}$$

is satisfied. If $f(z) \equiv \cos(\lambda z + \alpha)$, α real, then the equality holds along certain lines parallel to the imaginary axis.

THEOREM 2. *If $f(z)$ satisfies the conditions of Theorem* 1, *then*

$$|f(z)| \leqq \cosh \lambda y \tag{6}$$

and, unless $f(z)$ is of the form $\cos(\lambda z + \alpha)$, the equality can occur only on the real axis.

PROOF. It will be sufficient to show that (6) is true on the imaginary axis. Suppose $f(z)$ is not of the form $\cos(\lambda z + \alpha)$, and for some y, $(|y| > 0)$, we have

$$|f(iy)| \geqq \cosh \lambda y.$$

From the expansion

$$\cos(\lambda i y + \beta) = \cos \beta \cosh \lambda y - i \sin \beta \sinh \lambda y$$

we see that $\cos(\lambda i y + \beta)$ has, for a suitable real β, the same amplitude as $f(iy)$. Since $|\cos(\lambda i y + \beta)| \leqq \cosh \lambda y \leqq |f(iy)|$, there is a real γ, $(0 < \gamma \leqq 1)$, such that $|\gamma f(iy)| = |\cos(\lambda i y + \beta)|$. Then

$$\cos(\lambda z + \beta) = \gamma f(z)$$

at the point $z = iy$, since the amplitudes and magnitudes are the same. But $\gamma f(z)$ satisfies the conditions of Theorem 1, so $\cos(\lambda z + \beta) - \gamma f(z)$ can have only real zeros. The contradiction proves Theorem 2.

* E. C. Titchmarsh, loc. cit., p. 119.

It has been shown by S. Bernstein* that if $f(z)$ satisfies the conditions of Theorem 1, then

$$|f'(z)| \leqq \lambda$$

on the real axis. Szegö (for the case in which $f(z)$ is a trigonometric polynomial) and later Boas (under essentially the same conditions as in Theorem 1) obtained† the stronger inequality

$$\{f'(z)\}^2 + \lambda^2\{f(z)\}^2 \leqq \lambda^2 \tag{7}$$

on the real axis. The authors obtained a generalization of (7) for complex values of z in a previous paper. Using Theorem 1 we can now prove this corollary:

COROLLARY. *If $f(z)$ satisfies the conditions of Theorem* 1, *then* (7) *is true on the real axis.*

PROOF. Suppose $f(z)$ satisfies the conditions of Theorem 1 and is not of the form $\cos(\lambda z+\alpha)$. At points of the real axis where $f(z)=\pm 1$ we must have $f'(z)=0$, so (7) is certainly true. Hence suppose that, at some point z_0, $|f(z_0)|<1$, and (7) is not satisfied. Then by suitable choice of real γ, $(0<\gamma\leqq 1)$, we have the equality

$$\{\gamma f'(z)\}^2 + \lambda^2\{\gamma f(z)\}^2 = \lambda^2. \tag{8}$$

Then since $\cos(\lambda z+\alpha)$ satisfies the differential equation

$$\left[\frac{d}{dz}\cos(\lambda z+\alpha)\right]^2 + \lambda^2[\cos(\lambda z+\alpha)]^2 = \lambda^2,$$

there is a real α so that, at the point z_0,

$$\cos(\lambda z+\alpha) = \gamma f(z), \qquad \frac{d}{dz}\cos(\lambda z+\alpha) = \gamma f'(z).$$

Thus the function

$$\cos(\lambda z+\alpha) - \gamma f(z)$$

has a double zero at the point z_0, where $|\gamma f(z_0)|<1$; but by Theorem 1 this is impossible. The contradiction proves the corollary.

Using Theorem 2 we prove a second theorem‡ of S. Bernstein.

* S. Bernstein, Comptes Rendus, vol. 176 (1923), p. 1603.

† G. Szegö, Schriften der Königsberger Gelehrten Gesellschaft, Naturwissenschaftliche Klasse, vol. 5 (1928), p. 69; R. P. Boas, Transactions of this Society, vol. 40 (1936), p. 287. See also Van der Corput and Schaake, Compositio Mathematica, vol. 2 (1935), p. 321; R. J. Duffin and A. C. Schaeffer, this Bulletin, vol. 43 (1937), p. 554.

‡ S. Bernstein, Communications de la Société Mathématique de Charkow, (2), vol. 14. See also M. Riesz, Acta Mathematica, vol. 40 (1916), p. 337.

BERNSTEIN'S THEOREM. *Let $P(s)$ be a polynomial of degree n or less with real coefficients such that in the interval $(-1, 1)$ of the real axis $|P(s)| \leqq 1$. If A and B are the semi-axes of an ellipse passing through the point s and having foci at the points 1 and -1, then*

$$|P(s)| \leqq (A+B)^n. \tag{9}$$

We shall prove that if $P(s)$ satisfies the conditions of Bernstein's theorem, then the stronger inequality

$$|P(s)| \leqq \tfrac{1}{2}\{(A+B)^n + (A+B)^{-n}\} \tag{10}$$

is satisfied.

PROOF. If $P(s)$ satisfies the conditions of Bernstein's theorem it is clear that $P(\cos z)$ is a polynomial of degree n in $\cos z$ and is bounded by 1 on the real axis, so it satisfies the conditions of Theorem 2 with $\lambda = n$. Then we have

$$|P(\cos z)| \leqq \cosh ny, \tag{11}$$

where $z = x+iy$. Let $s = \sigma + it$ be any fixed point not in the interval $(-1, 1)$ of the real axis, and choose a z so that $s = \cos z$. Then we have the relations

$$\sigma = \cos x \cosh y,$$

$$t = -\sin x \sinh y,$$

and on eliminating x we obtain the equation

$$\frac{\sigma^2}{A^2} + \frac{t^2}{B^2} = 1,$$

where

$$A = \cosh|y|, \qquad B = \sinh|y|.$$

Thus the point s lies on an ellipse with center at the origin and semi-axes A and B, and the foci are at the points 1 and -1 since $A^2 - B^2 = 1$. Since $A+B = e^{|y|}$ we see that

$$\cosh ny = \tfrac{1}{2}\{(A+B)^n + (A+B)^{-n}\},$$

and on putting this in (11) we have (10), namely,

$$|P(s)| = |P(\cos z)| \leqq \tfrac{1}{2}\{(A+B)^n + (A+B)^{-n}\}.$$

This is a "best possible" inequality in the sense that if $P(s)$ is the nth Tchebycheff polynomial, $T_n(s) = \cos(n \cos^{-1} s)$, then (10) becomes an equality along certain lines in the complex plane.

PURDUE UNIVERSITY

APPROXIMATION TO A GIVEN FUNCTION BY MEANS OF POLYNOMIALS IN ANOTHER GIVEN FUNCTION*

W. C. RISSELMAN

1. **Introduction.** The problem considered in this note is that of approximating to a given function $f(x)$ in a given finite interval by means of polynomials in another given function $\phi(x)$ so as to minimize the integral of a positive power of the absolute value of the error. The purpose is to extend the results of Jackson in the theory of approximation by means of polynomials to the case in which x is replaced by $\phi(x)$. Questions of convergence are confined to uniform convergence. While some of the facts in connection with the problem can be recognized almost immediately, further study leads presently to material complications. This note will give some of the results which are obtained most readily, and others which by their limitations indicate the directions in which difficulty is encountered. It contains the first steps toward the solution of a general problem whose interest is believed to be so great as to warrant an approach under somewhat strong restrictions. While some of the hypotheses seem artificial they are made because of the necessity of recognizing definite limits beyond which the problem does not appear to lend itself to treatment by the methods employed. The study of the problem of convergence is divided into two sections. First is considered the case in which $\phi(x)$ is monotone; then, the case in which $\phi(x)$ is not monotone.

2. **Existence and uniqueness.** The following theorem on existence is a corollary of a theorem proved by Jackson.†

THEOREM 1. *Let $\phi(x)$ be a real function of the real variable x which has the following properties:*

A. *$\phi(x)$ is bounded and measurable for $a \leqq x \leqq b$.*

B. *If the set where ϕ assumes any n values is excluded, the measure of the complementary set is positive.*

Then if $f(x)$ is bounded and measurable on (a, b), and if m is a positive constant, there exists at least one set of coefficients c in

$$P_n[\phi(x)] = c_{0n} + c_{1n}\phi(x) + c_{2n}\phi^2(x) + \cdots + c_{nn}\phi^n(x)$$

* Presented to the Society, April 7, 1928.

† *A generalized problem in weighted approximation*, Transactions of this Society, vol. 26 (1924), p. 137.

for which the value of the integral

$$\int_a^b |f(x) - P_n[\phi(x)]|^m dx \tag{1}$$

is a minimum. If $m>1$, the minimizing polynomial is uniquely determined.

A polynomial which has the minimizing property will be called an approximating polynomial.

3. $\phi(x)$ **monotone.** A theorem on convergence of a sequence of approximating polynomials is the following:

THEOREM 2. *If $f(x)$ is defined and has a continuous pth derivative on the interval (a, b), and if $\phi(x)$ is defined and has p continuous derivatives on (a, b), the first of which is non-vanishing, then the sequence of approximating polynomials, or any sequence of approximating polynomials if the determination is not unique, converges uniformly to $f(x)$ on (a, b) as $n\to\infty$ if m satisfies the condition $m\geqq 2/p$.*

If $\phi'(x)>0$, if $\phi(a)=A$ and $\phi(b)=B$, and if the hypotheses of the theorem are satisfied, the integral (1) becomes

$$\int_A^B g'(y)\,|h(y) - P_n(y)|^m dy, \tag{2}$$

where $g(y)$ is the function defined by the equation $y=\phi(x)$, and $h(y)=f[g(y)]$. The function $g'(y)$ is continuous and greater than zero on (A, B) and $h(y)$ has a continuous pth derivative. The proof then makes use of a corollary to Bernstein's theorem on the derivative of a real polynomial, or Markoff's theorem on the derivative of a polynomial, together with a theorem on polynomial representation.*

If $\phi(x)$ is monotone and $\phi'(x)$ has zeros, the following theorem holds:

THEOREM 3. *If $f(x)$ is defined and has a continuous first derivative on (a, b), if $\phi(x)$ is a monotone function which is defined and has a continuous first derivative which has a finite number of zeros on this interval, and if $f'(x)/\phi'(x)$ has a finite limit as x approaches a zero of $\phi'(x)$, the sequence of approximating polynomials $\{P_n[\phi(x)]\}$ converges uniformly to $f(x)$ on (a, b) as $n\to\infty$, provided $m\geqq 2$.*

* For details the reader is referred to pp. 96 and 97 and the corollary on p. 18 of *The Theory of Approximation*, by D. Jackson, American Mathematical Society Colloquium Publications, vol. 11, New York, 1930. Hereafter this book will be called J.

Under these hypotheses the weight function $g'(y)$ in (2) has discontinuities. However, the validity of the transformation of (1) into the form of (2) can be established by examining the formulas with consideration of the fact that $g(y)$ is continuous and is the integral of its own derivative. Then one proves that $h'(y)$ is a continuous function. Let (D, E) be an interval in which $g'(y)$ is continuous at every point except D. Let $\lim_{x \to g(D)} f'(x)/\phi'(x) = L$. Let $\psi(y) = h'(y)$ on $D < y \leqq E$, $\psi(D) = L$, and

$$H(y) = h(E) + \int_E^y \psi(y)dy.$$

Then $H'(y)$ is continuous on the closed interval (D, E), and $H(y) = h(y)$ on this closed interval. It can now be seen that the hypotheses of one of Jackson's theorems are satisfied.*

It is to be noticed that the above hypothesis that $f'(x)/\phi'(x)$ has a finite limit as x approaches a zero of $\phi'(x)$ is very restrictive. It means, for instance, that f' vanishes wherever ϕ' vanishes. But if $f'/\phi' \to \infty$ as x approaches a zero of ϕ', then the first derivative of $h(y)$, the function to be approximated, is discontinuous.

The hypothesis that the derivative of $\phi(x)$ is defined everywhere will now be abandoned. The following theorem holds:

THEOREM 4. *If $\phi(x)$ is a monotonic function which is defined for $a \leqq x \leqq b$ and satisfies the condition that there exist positive numbers k_1 and k_2 independent of x such that*

$$k_1 |x_2 - x_1|^\beta \leqq \left| \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \right| \leqq k_2,$$

where $a \leqq x_1$, $x_2 \leqq b$, $x_1 \neq x_2$, and $\beta \geqq 0$ and independent of x; if $f(x)$ is any function which on (a, b), is defined and satisfies a Lipschitz condition of order α $|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|^\alpha$, where λ is independent of x, then the sequence of approximating polynomials $\{P_n[\phi(x)]\}$ converges uniformly to $f(x)$ on (a, b) as $n \to \infty$ provided α, β, and m satisfy $\alpha/(1+\beta) > 2/m$, and it converges uniformly to $f(x)$ on any closed interval interior to (a, b) as $n \to \infty$ provided that these quantities satisfy the inequality $\alpha/(1+\beta) > 1/m$.

Under these hypotheses the integral (1) equals the Stieltjes integral

$$\gamma_n = \int_A^B |f[g(y)] - P_n(y)|^m dg.$$

* J, Theorem VII of chap. 3.

Suppose there exists a sequence of polynomials $\{p_n(y)\}$ such that $|f[g(y)]-p_n(y)| \leqq \epsilon_n$ on (A, B). By using the corollary to Bernstein's theorem which was used before and an iteration process* it can be shown that $|f[g(y)]-P_n(y)|$ does not exceed

$$5\epsilon_n + d_1 n^{2/m}\gamma_n^{1/m} \tag{3}$$

on (a, b) and

$$5\epsilon_n + d_2(n^\nu\gamma_n)^{1/m} \tag{4}$$

on any closed interval interior to (a, b), where $\nu = 1+(m+1)^{-k}$, k is an arbitrary positive integer, and the d's are constants. In the following manner it is seen that $f[g(y)]$ satisfies a Lipschitz condition of order $\alpha/(1+\beta)$. Since $f(x)$ satisfies the Lipschitz condition of order α, if y_1 and y_2 are any two values of y on (A, B),

$$|f[g(y_2)] - f[g(y_1)]| \leqq \lambda |g(y_2) - g(y_1)|^\alpha.$$

From one of the conditions imposed on ϕ,

$$|g(y_2) - g(y_1)|^\alpha \leqq (1/k_1)^{\alpha/(1+\beta)} |y_2 - y_1|^{\alpha/(1+\beta)}.$$

Therefore $f[g(y)]$ satisfies the Lipschitz condition

$$|f[g(y_2)] - f[g(y_1)]| \leqq \lambda_1 |y_2 - y_1|^{\alpha/(1+\beta)},$$

where $\lambda_1 = \lambda(1/k_1)^{\alpha/(1+\beta)}$. Since $f[g(y)]$ satisfies this Lipschitz condition, ϵ_n may be taken equal to $K/n^{\alpha/(1+\beta)}$, where K is a constant, and γ_n is found to be less than a constant times $n^{-m\alpha/(1+\beta)}$. These expressions are then substituted in (3) and (4).

If $\alpha = 1$ and $\beta = 0$, the last theorem reduces to the theorem:

If $\phi(x)$ and $f(x)$ are defined for $a \leqq x \leqq b$, if $\phi(x)$ is monotone, and if there exist positive constants k_1, k_2, and λ independent of x such that

$$k_1 \leqq \left|\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}\right| \leqq k_2, \qquad \left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| \leqq \lambda,$$

where x_1 and x_2 are any two distinct values of x on (a, b), then the sequence of approximating polynomials converges uniformly to $f(x)$ on (a, b) as $n \to \infty$ if $m > 2$, and it converges uniformly to $f(x)$ on any closed interval interior to (a, b) as $n \to \infty$ if $m > 1$.

Under suitable hypotheses,† including the one that $\phi(x)$ has a con-

* J, pp. 96 and 97, and J. M. Earl, *Polynomials of best approximation on an infinite interval*, Transactions of this Society, vol. 32 (1930), p. 3.

† The reader will find that a theorem can be formulated so that its proof depends on Theorem VII in chap. 3 of J.

tinuous non-vanishing first derivative where ϕ is continuous, it is also found that the sequence of approximating polynomials in a monotone $\phi(x)$ converges uniformly to $f(x)$ on (a, b) as $n \to \infty$ if $m \geqq 2$ in case $\phi(x)$ has a finite number of finite jumps. In this case the integral is transformed to the form (2). If $\phi(x)$ jumps from C to D at a certain value of x, $h(y)$ is taken to be a suitable polynomial of the third degree in y on (C, D), and the hypotheses on f and ϕ may be made so that $h(y)$ has a continuous first derivative on (A, B). The weight function is taken to be zero on the open interval (C, D).

4. $\phi(x)$ **not monotone.*** If $\phi(x)$ has the same value for different values of x, so also will any polynomial in $\phi(x)$ for these same values of x. Now, if the values of $f(x)$ for these values of x are different, no sequence $\{P_n[\phi(x)]\}$ of approximating polynomials can converge to $f(x)$ for these values of x. However, under certain hypotheses it can be proved that such a sequence converges to a suitably defined mean of the values of $f(x)$ corresponding to a single value of $\phi(x)$.

Let $f(x)$ be a function which is defined and continuous and which has a continuous first derivative on $a \leqq x \leqq b$, and let $\phi(x)$ be a function which satisfies the following conditions:

A. It is continuous on $a \leqq x \leqq b$.

B. It has a finite number, say $(q+1)$, of maxima and minima on (a, b).

C. The maxima of ϕ are all equal and the minima are all equal.

D. Let the values of x, arranged in order of increasing values, for which ϕ has maxima or minima be denoted by $a = a_0, a_1, a_2, \cdots, a_q = b$. Then $\phi(x)$ has continuous first and second derivatives on all intervals $a_i \leqq x \leqq a_{i+1}$, $(i = 0, 1, \cdots, q-1)$.

E. The relation $|\phi'(x)| \geqq k > 0$, where k is independent of x, holds for all values of x for which $|\phi'(x)|$ is defined.

The integral (1) may be written in the form

$$\gamma_n = \sum_{i=1}^{q} \int_A^B |g_i'(y)| \, |f[g_i(y)] - P_n(y)|^m \, dy,$$

where $g_i(y)$ is the function inverse to $\phi(x)$ in the ith subinterval and A is the minimum value of ϕ and B its maximum value.

Consider the function

$$H(z, y) = \sum_{i=1}^{q} |g_i'(y)| \, |z - f[g_i(y)]|^m.$$

* A paper on a different subject, the methods of which nevertheless have a bearing on the rest of this note, is that by D. Jackson entitled *On the trigonometric representation of an ill-defined function*, Annals of Mathematics, (2), vol. 26 (1924), pp. 8–20.

If any value on (A, B) is assigned to y there is exactly one value of z which minimizes $H(z, y)$ if $m>1$. If there were two such values, say z_1 and z_2, then since the curve $Y=|X|^m$ is concave upward,

$$\left|\frac{z_1+z_2}{2}-f(g_i)\right|^m<\tfrac{1}{2}\{|z_1-f(g_i)|^m+|z_2-f(g_i)|^m\},$$

$$i=1, 2, \cdots, q,$$

and a contradiction would be obtained. Thus z is defined as a single-valued function of y, which will be denoted by $F(y)$. By taking the partial derivative of H with respect to z it is found that the equation

$$(5)\qquad \sum_{i=1}^{q}|g_i'(y)|\,|z-f[g_i(y)]|^{m-1}\operatorname{sgn}\{z-f[g_i(y)]\}=0,$$

defines z as this function of y. Under suitable hypotheses it will be proved that the sequence $\{P_n(y)\}$ converges uniformly to $F(y)$ as $n\to\infty$.

Consider, first, two special cases: (i) $m=2$, q is arbitrary; (ii) $m>2$, $q=2$, where the explicit expression for $F(y)$ will be given.

(i). *$m=2$, q is arbitrary.* If $m=2$, it is found from (5) that

$$F(y)=\frac{\sum_{i=1}^{q}|g_i'(y)|\,f[g_i(y)]}{\sum_{i=1}^{q}|g_i'(y)|}.$$

The integral γ_n may be expressed in the form

$$\gamma_n=\int_A^B\sum_{i=1}^{q}|g_i'(y)|\,\{(f[g_i(y)]-F(y))+(F(y)-P_n(y))\}^2dy.$$

By expanding this expression and taking (5) into account it is found that

$$\gamma_n=\int_A^B\sum_{i=1}^{q}|g_i'|\,[f(g_i)-F]^2dy+\int_A^B\left\{\sum_{i=1}^{q}|g_i'|\right\}\{F-P_n\}^2dy.$$

Thus the original problem is reduced to that of approximating to $F(y)$ by means of polynomials in y. From the hypotheses it follows that $F(y)$ has a continuous first derivative on (A, B) and that the weight function $\sum_{i=1}^{q}|g_i'|$ has a positive minimum on this interval. Thus one has the theorem:*

* See J, Theorem VII of chap. 3.

THEOREM 5. *If $f(x)$ has a continuous derivative on (a, b), if $\phi(x)$ satisfies the hypotheses* A *to* E, *and if $m=2$, the sequence $\{P_n[\phi(x)]\}$ converges uniformly as $n\to\infty$ to the function*

$$F[\phi(x)] = \frac{\sum_{i=1}^{q} |\, g_i'(\phi)\,|\, f[g_i(\phi)]}{\sum_{i=1}^{q} |\, g_i'(\phi)\,|} \quad \text{on} \quad (a, b),$$

where $F[\phi(x)]$ is obtained as follows: Corresponding to any value of x on (a, b) a value of y is determined by means of $y=\phi(x)$. Using this value of y one obtains from $x=g_i(y)$, $(i=1, 2, \cdots, q)$, q values of x, say, $x^{(1)}, x^{(2)}, \cdots, x^{(q)}$. Then $F[\phi(x)]$ is the indicated mean of the values $f(x^{(1)}), f(x^{(2)}), \cdots, f(x^{(q)})$.

(ii) $m>2$, $q=2$. If the inverse of ϕ is double-valued and $m>2$, the function of y defined by (5) is given by

$$F(y) = \frac{|\, g_1'\,|^{1/(m-1)} f(g_1) + |\, g_2'\,|^{1/(m-1)} f(g_2)}{|\, g_1'\,|^{1/(m-1)} + |\, g_2'\,|^{1/(m-1)}}. \tag{6}$$

Again $F(y)$ has a continuous first derivative on (A, B). Let the function $G_2(z, y)$ be defined by the equation

$$G_2(z, y) = \sum_{i=1}^{2} |\, g_i'(y)\,|\, |\, z - \{f[g_i(y)] - p_n(y)\}\,|^m,$$

where $p_n(y)$ is the Tchebychef polynomial, of degree $\leqq n$, of best approximation to $F(y)$ on (A, B). Let $u_i(y)=f[g_i(y)]-p_n(y)$ and let $\Pi_n(y)=P_n(y)-p_n(y)$. Then $u_i(y)-\Pi_n(y)=f[g_i(y)]-P_n(y)$ and $\gamma_n=\int_A^B G_2[\Pi_n(y), y]dy$. The second derivative of G_2 with respect to z is continuous in the region $R: (-M\leqq z\leqq M;\ A\leqq y\leqq B)$, where M denotes the maximum of $|\,F(y)\,|$ on (A, B). Let $z_0=F(y)-p_n(y)$ be the value of z for which $G_2(z, y)$ is a minimum. Expansion of G_2 according to Taylor's theorem gives

$$G_2(0, y) = G_2(z_0, y) + \tfrac{1}{2} z_0^2 G_2''(\zeta_0, y),$$

where ζ_0 is between 0 and z_0, and G_2'' denotes the second derivative with respect to z. Let ϵ_n be the maximum of $|z_0|$ on (A, B). Then $\epsilon_n\leqq M$. It follows that ζ_0 satisfies $-M\leqq\zeta_0\leqq M$ for all y on (A, B). Let the maximum of G_2'' in R be denoted by $2V$. Then

$$G_2(0, y) \leqq G_2(z_0, y) + z_0^2 V$$

for all y on (A, B). Therefore

$$\gamma_n = \int_A^B G_2(\Pi_n, y)dy \leqq \int_A^B G_2(0, y)dy$$
$$\leqq \int_A^B G_2(z_0, y)dy + V\epsilon_n^2(B - A). \tag{7}$$

Let $\mu_n = |\Pi_n(y_0)|$ be the maximum of $|\Pi_n(y)|$ on (A, B). By using Markoff's theorem and the mean value theorem it is found in the usual manner that, if $\mu \geqq 4\epsilon_n$, then $|\Pi_n - z_0| \geqq \mu_n/4$ throughout an interval L of length $(B-A)/4n^2$. Let K be the maximum and k the minimum of $|g_1'(y)| + |g_2'(y)|$ on (A, B). Let $d > 0$ be a fixed number which satisfies $d < k^{1/m}K^{-1/m}/4(1+k^{1/m}K^{-1/m})$. The part of the interval L on which

$$|f[g_1(y)] - f[g_2(y)]| \leqq d\mu_n$$

is considered separately from the rest on which this expression is greater than $d\mu_n$. In the first case, use is made of the explicit expressions for $G_2(\Pi_n, y)$ and $G_2(0, y)$, and in the second case use is made of the expansion

$$G_2(\Pi_n, y) = G_2(z_0, y) + \tfrac{1}{2}(\Pi_n - z_0)^2 G_2''(\zeta, y),$$

in order to show that $G_2(\Pi_n, y) \geqq G_2(z_0, y) + c\mu_n^m$ throughout L, where c is constant and positive. Then it is seen that

$$\gamma_n \geqq \int_L [G_2(z_0, y) + c\mu_n^m]dy + \int_{cL} G_2(z_0, y)dy$$
$$= \int_A^B G_2(z_0, y)dy + c\mu_n^m \frac{B - A}{4n^2}, \tag{8}$$

where $\int_L$ means the integral over the interval L, and $\int_{cL}$ means the integral over the rest of (A, B). From (7) and (8) it follows that

$$\mu_n \leqq \left(\frac{4V}{c}\right)^{1/m} (n\epsilon_n)^{2/m} \quad \text{if} \quad \mu_n \geqq 4\epsilon_n.$$

Now $|F(y) - P_n(y)| \leqq \epsilon_n + \mu_n$ on $(A\ B)$. Since $F(y)$ has a continuous derivative, $\lim_{n\to\infty} n\epsilon_n = 0$. Therefore, $|F(y) - P_n(y)|$ approaches zero uniformly, whether $\mu_n \geqq 4\epsilon_n$ or not. One now has the theorem:

THEOREM 6. *If $f(x)$ is defined and has a continuous first derivative on (a, b) and if $\phi(x)$ satisfies the hypotheses* A *to* E *with $q = 2$, the sequence $\{P_n[\phi(x)]\}$ converges uniformly to $F[\phi(x)]$ on (a, b) as $n \to \infty$ if $n > 2$, where F is given by* (6).

Regarding the general case in which $m>2$ and $q\geqq 3$ one can state the following theorem:

THEOREM 7. *If $f(x)$ is a univariant function which is defined on (a, b) and has a continuous first derivative on this interval, and if $\phi(x)$ satisfies hypotheses* A *to* E *with $q\geqq 3$, the sequence $\{P_n[\phi(x)]\}$ converges uniformly to $F[\phi(x)]$ on (a, b) as $n\to\infty$ if $m>2$, where F is the function defined implicitly by* (5).

In this case no explicit expression is obtained for $F(y)$. The function

$$G(z, y) = \sum_{i=1}^{q} | g_i'(y) | \, | z - \{f[g_i(y)] - p_n(y)\} |^m,$$

where $p_n(y)$ is the Tchebychef polynomial corresponding to $F(y)$, is introduced. The hypothesis of univariance of $f(x)$ is used to prove that $F'(y)$ is continuous on (A, B) and to show that $G''(z, y)$ has a positive lower bound in the region R': $(A\leqq y\leqq B;\ -\infty<z<\infty)$. The expansion

$$G(z, y) = G(z_0, y) + \tfrac{1}{2}(z - z_0)^2 G''(\zeta, y)$$

is used to compare $G(0, y)$ and $G(\Pi_n, y)$ with $G(z_0, y)$. The rest of the proof is similar to the corresponding parts of the proof of the theorem immediately preceding this one.

In case $m=1$, it can be shown, by methods that offer no essential novelty, that the sequence of approximating polynomials converges uniformly to $F[\phi(x)]$ in any finite set of closed intervals not containing points where ϕ has maxima or minima, provided f and ϕ satisfy suitable hypotheses under which ϕ has a continuous first derivative everywhere on (a, b), this derivative not being required to be everywhere different from zero. However, these hypotheses exclude functions ϕ with a continuous second derivative. In special cases it is found that convergence occurs even in this case. For example, if the interval is $(-1, 1)$, then $P_n(x^2)$ is the partial sum of the Legendre series for $E(x)=\frac{1}{2}[f(x)+f(-x)]$ taken through the term involving $\phi_{2n}(x)$, where $\phi_i(x)$ is the ith normalized Legendre polynomial.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

A QUADRATIC FORM PROBLEM IN THE CALCULUS OF VARIATIONS*

A. ADRIAN ALBERT

The problem which we shall discuss arose in connection with sufficiency theorems in the multiple integral problem of the calculus of variations. It was proposed by Professor G. A. Bliss to his University of Chicago seminar (summer, 1937) and communicated to the author by Professor W. T. Reid. The result of the author's investigation presented here is a very interesting theorem on real quadratic forms.

We first have the trivial lemma:

LEMMA 1. *Let f and g be real quadratic forms in $x_1, \cdots, x_n$, and g be negative definite. Then there exists a real non-singular linear transformation carrying g and f respectively into*

$$G = -(x_1^2 + \cdots + x_n^2), \qquad F = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2,$$

where the λ_i are the roots of the determinant $|f+\lambda g| = 0$† and may be arranged so that

$$\lambda_n \geqq \lambda_{n-1} \geqq \cdots \geqq \lambda_1. \tag{1}$$

Moreover $f+\lambda g$ is positive definite if and only if

$$\lambda_1 > \lambda > -\infty. \tag{2}$$

For we may carry g into G. Apply a real orthogonal transformation carrying the resulting f into diagonal form F. The λ_i are clearly the roots of $|F+\lambda G| = 0$ and hence of $|f+\lambda g| = 0$. Finally $f+\lambda g$ is positive definite if and only if $F+\lambda G$ is positive definite, that is if $\lambda_i - \lambda \geqq \lambda_1 - \lambda > 0$.

We next derive the following lemma:

LEMMA 2. *Let g be non-singular and indefinite of index p, and let there exist a real λ_0 such that $f+\lambda_0 g = h$ is positive definite. Then there exists a real non-singular linear transformation carrying g and f respectively into*

$$\begin{aligned} G &= x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_n^2), \\ F &= -(\lambda_1 x_1^2 + \cdots + \lambda_p x_p^2) + \lambda_{p+1} x_{p+1}^2 + \cdots + \lambda_n x_n^2, \end{aligned} \tag{3}$$

* Presented to the Society, November 27, 1937.

† By this we mean the determinant of the matrix corresponding to the pencil of forms $f+\lambda g$.

with λ_i the roots of $|f+\lambda g|=0$. Moreover

$$\lambda_j > \lambda_i, \qquad i = 1, \cdots, p; j = p+1, \cdots n, \tag{4}$$

so that we may arrange the roots in the order

$$\lambda_n \geqq \lambda_{n-1} \geqq \cdots \geqq \lambda_{p+1} > \lambda_p \geqq \lambda_{p-1} \geqq \cdots \geqq \lambda_1. \tag{5}$$

Finally $f+\lambda g$ is positive definite if and only if

$$\lambda_{p+1} > \lambda > \lambda_p. \tag{6}$$

For as in Lemma 1 we carry h to $\xi_1^2 + \cdots + \xi_n^2$ and then apply an orthogonal transformation leaving h unaltered and carrying g into $\sum_{i=1}^n \delta_i \xi_i^2$ for real $\delta_i \neq 0$. Let $x_i = |\delta_i|^{1/2}\xi_i$; then $g = \sum_{i=1}^n \pm x_i^2$, $h = \sum_{i=1}^n |\delta_i|^{-1} x_i^2$. The index of g is an invariant and g may then be carried into G. But h is diagonal and hence so is $f = h - \lambda_0 g$. Write $f = F$ as in (3); then $F + \lambda G = \sum_{i=1}^p (\lambda - \lambda_i) x_i^2 + \sum_{j=p+1}^n (\lambda_j - \lambda) x_j^2$ so that the λ_i are the characteristic roots of $|f+\lambda g| = 0$. The value $x_k = 0$ for $k \neq i$ and j, $x_i = x_j = 1$, i and j as in (4), gives $g = 0$, $h = f + \lambda_0 g = f = \lambda_j - \lambda_i > 0$, and we have (4). Then $F + \lambda G$, and hence $f + \lambda g$, is positive definite if and only if $\lambda_j > \lambda > \lambda_i$, which is satisfied if and only if (6) holds.

Observe that Lemma 1 becomes a special case of Lemma 2 if we allow $p = 0$ and $\lambda_0 = -\infty$.

THEOREM. *Let f and g be real quadratic forms in $x_1, \cdots, x_n$, and let f be positive for all real x_i not all zero such that $g = 0$. Then there exists a real number λ such that $f + \lambda g$ is positive definite.*

The properties of the theorem are clearly invariant under real non-singular linear transformations, and under replacement of g by μg for μ real and not zero. We shall use such transformations.

The result is true for $n = 1$ since then either $g = 0$ and f is necessarily positive definite, or we may take $g = x^2$, $f = ax^2$, $f + (1-a)g = x^2$ positive definite. We thus make an induction and assume our theorem for forms in $n-1$ variables.

If the rank of g were $r < n$, we could take g to be a form in $x_1, \cdots, x_r$. Then $x_1 = \cdots = x_r = 0$ in f gives a form $f_2 = f_2(x_{r+1}, \cdots, x_n) > 0$ for all $x_{r+1}, \cdots, x_n$ not all zero. Hence f_2 is definite and may be taken to be $x_{r+1}^2 + \cdots + x_n^2$ by a transformation on $x_{r+1}, \cdots, x_n$ not altering g. Thus $f = x_{r+1}^2 + \cdots + x_n^2 + 2\sum_{i=r+1}^n x_i L_i + f_1(x_1, \cdots, x_r)$, where the L_i are linear forms in $x_1, \cdots, x_r$. The transformation

$$X_i = x_i + L_i, \quad X_j = x_j, \qquad j = 1, \cdots, r; i = r+1, \cdots, n,$$

does not alter the form of g and carries f into $X_{r+1}^2+\cdots+X_n^2+F(X_1,\cdots,X_r)$, g into $G(X_1,\cdots,X_r)$. Put $X_{r+1}=\cdots=X_n=0$ and have $F>0$ for all X_j not all zero such that $G=0$. Then $F+\lambda G$ is positive definite for a real λ and so is $X_{r+1}^2+\cdots+X_n^2+F+\lambda G=f+\lambda g$.

Hence let g be non-singular. Our result follows from Lemma 1 if g is definite. Now let g be indefinite and apply a linear transformation carrying g into the form (3). Then $f_0=f(0, x_2, \cdots, x_n)>0$ for all $x_2, \cdots, x_n$ not all zero and making $g_0=g(0, x_2, \cdots, x_n)=0$. By the hypothesis of our induction we may apply Lemma 2 and write either

$$f_0 = -(\lambda_2 x_2^2+\cdots+\lambda_p x_p^2)+\lambda_{p+1}x_{p+1}^2+\cdots+\lambda_n x_n^2, \tag{7}$$

where

$$\lambda_n \geqq \lambda_{n-1} \geqq \cdots \geqq \lambda_{p+1} > \lambda_p \geqq \lambda_{p-1} \geqq \cdots \geqq \lambda_2, \tag{8}$$

or

$$p=1, \qquad f_0=\lambda_2 x_2^2+\cdots+\lambda_n x_n^2, \qquad \lambda_n \geqq \cdots \geqq \lambda_2. \tag{9}$$

In the respective cases we take λ in the intervals

$$\lambda_{p+1} > \lambda > \lambda_p, \qquad \lambda_2 > \lambda > -\infty. \tag{10}$$

Now

$$f = f_0 + 2\sum_{i=2}^{n} b_i x_i x_1 + a x_1^2 \tag{11}$$

so that

$$\begin{aligned} f+\lambda g = &\sum_{i=2}^{p}(\lambda-\lambda_i)\left(x_i+\frac{b_i x_1}{\lambda-\lambda_i}\right)^2 \\ &+\sum_{j=p+1}^{n}(\lambda_j-\lambda)\left(x_j+\frac{b_j x_1}{\lambda_j-\lambda}\right)^2+x_1^2\phi(\lambda), \end{aligned} \tag{12}$$

where

$$\phi(\lambda) = a+\lambda-\left(\sum_{i=2}^{p}\frac{b_i^2}{\lambda-\lambda_i}+\sum_{j=p+1}^{n}\frac{b_j^2}{\lambda_j-\lambda}\right). \tag{13}$$

The values

$$x_i = -\frac{b_i}{\lambda-\lambda_i}, \qquad x_j = -\frac{b_j}{\lambda_j-\lambda}, \qquad x_1 = 1, \tag{14}$$

$$i = 2, \cdots, p; \; j = p+1, \cdots, n,$$

make

$$(15)\quad f+\lambda g=\phi(\lambda),\ g=1+\sum_{i=2}^{p}\frac{b_i^2}{(\lambda-\lambda_i)^2}-\sum_{j=p+1}^{n}\frac{b_j^2}{(\lambda_j-\lambda)^2}=\phi'(\lambda),$$

a rational function of λ continuous in the interval (11).

If all the $b_k\neq 0$ we have $\phi'(\lambda_{p+1})=-\infty$, $\phi'(\lambda_p)=\infty$ if $p>1$, while if $p=1$, then $\phi'(-\infty)=1>0$. Hence there exists a λ in the intervals (10) such that $\phi'(\lambda)=g=0$. But then our hypothesis states that $f=\phi(\lambda)>0$. By (12), and since $\phi(\lambda)>0$, we have $f+\lambda g$ positive definite.

There remains the case where some $b_k=0$. Here we may permute the x_i and change the sign of g if necessary and carry the corresponding x_k into x_1. Then $f=-\lambda_1 x_1^2+f_0(x_2,\cdots,x_n)$. As in the proof above we may carry f_0 into (7) and have f in the form (3). But $f>0$ for $g=0$ and as in the proof of Lemma 2 we have (5), and $f+\lambda g$ is positive definite for λ as in (6).

We have proved our theorem. Notice that our reduction to the case g non-singular together with Lemmas 1, 2 determines the range of λ for which $f+\lambda g$ is positive definite.

THE UNIVERSITY OF CHICAGO

THE RIEMANNIAN CURVATURE OF A HYPERSURFACE*

AARON FIALKOW†

1. **Introduction.** It is a well known theorem of Gauss that the total curvature of any two dimensional surface in euclidean three space is equal to the product of the principal normal curvatures. Eisenhart‡ has shown that a generalization of this theorem applies to Riemann spaces of class one; that is, the hypersurfaces of an n-dimensional flat space. He proves the theorem:

When the lines of curvature of a Riemann space V_n of class one are real and none of them is tangent to a null vector, the Riemannian curvature at a point for the orientation determined by the direction of two lines of curvature at the point is numerically equal to the product of the corresponding normal curvatures; the sign is determined by the character of the normal to V_n in the enveloping flat V_{n+1}.

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† National Research Fellow.

‡ L. P. Eisenhart, *Riemannian Geometry*, 1926, p. 199.

This is Gauss' theorem for a flat V_{n+1}. The analogous theorem, which is true of the hypersurfaces of any Riemann space, appears to have been overlooked and is derived in §3 of this note. From this result, we establish the following theorem which is the converse of the theorem of Gauss for a flat V_{n+1}:

Let V_n be any hypersurface of a V_{n+1} such that the lines of curvature of V_n are real and none of them is tangent to a null vector. Let the Riemannian curvature of V_n at a point for the orientation determined by the direction of two lines of curvature at the point be numerically equal to the corresponding normal curvatures, the sign being determined by the character of the normal to V_n in the enveloping V_{n+1}. Then V_{n+1} is a flat space.

2. **Hypersurfaces of a V_{n+1}.** We begin by a short summary of those portions of the theory of hypersurfaces which are necessary in our work. Let V_{n+1} be a real Riemann space with the first fundamental form*

$$ds^2 = a_{\alpha\beta}dy^\alpha dy^\beta. \tag{1}$$

Any real hypersurface V_n immersed in V_{n+1} is defined by a system of equations $y^\alpha = y^\alpha(x^1, x^2, \cdots, x^n)$, where the rank of the Jacobian matrix $\|\partial y^\alpha/\partial x^i\|$ is n. The metric induced in the hypersurface by (1) determines the first fundamental form of V_n as

$$ds^2 = g_{ij}dx^i dx^j, \tag{2}$$

where

$$g_{ij} = a_{\alpha\beta}y^\alpha_{,i}y^\beta_{,j}.\dagger \tag{3}$$

We assume that real coordinate systems $\{y^\alpha\}$ and $\{x^i\}$ and an open region $\mathfrak{R}$ of the n-dimensional arithmetic number space $\{x^i\}$ exist such that $a_{\alpha\beta}(y^\gamma)$ are real functions of class C^2 and $y^\alpha(x^i)$ are real functions of class C^3 for $x \subset \mathfrak{R}$. Since the rank of $\|\partial y^\alpha/\partial x^i\|$ is n, (2) is non-singular for $x \subset \mathfrak{R}$ although it may be indefinite. Under these conditions the Gauss equations

$$R_{hijk} = e(\Omega_{hj}\Omega_{ik} - \Omega_{hk}\Omega_{ij}) + \overline{R}_{\alpha\beta\gamma\delta}y^\alpha_{,h}y^\beta_{,i}y^\gamma_{,j}y^\delta_{,k} \tag{4}$$

are satisfied for $x \subset \mathfrak{R}$. In these equations R_{hijk} and $\overline{R}_{\alpha\beta\gamma\delta}$ are the

* Throughout this paper, Greek indices have the range $1, 2, \cdots, n+1$ and Latin indices the range $1, 2, \cdots, n$. An index which appears twice in an expression is to be summed over the appropriate range unless the index appears in parentheses.

† The comma denotes covariant differentiation with respect to the tensor g_{ij}.

Riemann curvature tensors of V_n and V_{n+1} respectively, and e is plus one or minus one, being defined by

$$e = a_{\alpha\beta}\xi^\alpha\xi^\beta, \tag{5}$$

where the ξ^α are the components of the unit normal to V_n. The quantities Ω_{ij} are the coefficients of the second fundamental form of V_n

$$\Omega_{ij} = a_{\alpha\beta}\xi^\alpha y^\beta_{,ij} + [\beta\gamma, \alpha]_a y^\beta_{,i} y^\gamma_{,j}\xi^\alpha, \tag{6}$$

where the brackets are the Christoffel symbols of the first kind formed with respect to $a_{\alpha\beta}$ and evaluated for $x \subset \mathfrak{R}$. It is clear that the functions R_{hijk} and $\overline{R}_{\alpha\beta\gamma\delta}$ are of class C^0, Ω_{ij} of class C', and ξ^α of class C^2 for $x \subset \mathfrak{R}$.

The directions of the lines of curvature of V_n are given by the vectors* ${}_{(p)}\lambda^i$ which satisfy

$$(\Omega_{ij} - K_p g_{ij})\,{}_{(p)}\lambda^i = 0, \tag{7}$$

where K_p are the principal normal curvatures and are the roots of the determinant equation

$$|\,\Omega_{ij} - K g_{ij}\,| = 0. \tag{8}$$

If the elementary divisors of (8) are simple (as is always the case if (3) is definite), there is at least one orthogonal ennuple of unit vectors ${}_{(p)}\lambda^i$ which satisfy (7). When the elementary divisors are not simple, the tangents of some of the lines of curvature are null vectors. If (2) is indefinite, the principal normal curvatures need not be real even though V_n and V_{n+1} are both real Riemann spaces. The tangent of a line of curvature is real when and only when the corresponding principal normal curvature is real.†

3. **The theorem of Gauss for any Riemann space V_{n+1}.** We assume that ${}_{(1)}\lambda^i$ and ${}_{(2)}\lambda^i$ are two unit vectors which are tangent to real lines of curvature at a point P corresponding to $x \subset \mathfrak{R}$. If these directions and any $(n-2)$ real unit vectors orthogonal to both ${}_{(1)}\lambda^i$ and ${}_{(2)}\lambda^i$ are chosen as coordinate directions, it follows from the algebraic theory that at P

$$\begin{aligned} &g_{13} = g_{14} = \cdots = g_{1n} = 0, && {}_{(1)}\lambda^2 = {}_{(1)}\lambda^3 = \cdots = {}_{(1)}\lambda^n = 0, \\ &g_{23} = g_{24} = \cdots = g_{2n} = 0, && {}_{(2)}\lambda^1 = {}_{(2)}\lambda^3 = \cdots = {}_{(2)}\lambda^n = 0, \\ &\Omega_{1i} = K_1 g_{1i}, && \Omega_{2i} = K_2 g_{2i}. \end{aligned} \tag{9}$$

* Here p denotes the vector and i the component.

† T. J. I'A. Bromwich, *Quadratic Forms and their Classification by Means of Invariant Factors*, 1906, chaps. 3 and 4. Also cf. M. Bôcher, *Introduction to Higher Algebra*, 1929, p. 305, and Eisenhart, loc. cit., pp. 108–112.

We denote by k_{pq} the Riemannian curvature for the orientation determined by ${}_{(p)}\lambda^i$ and ${}_{(q)}\lambda^i$. By definition,

$$k_{12} = \frac{R_{hijk}{}_{(1)}\lambda^h{}_{(2)}\lambda^i{}_{(1)}\lambda^j{}_{(2)}\lambda^k}{(g_{hj}g_{ik} - g_{hk}g_{ij})_{(1)}\lambda^h{}_{(2)}\lambda^i{}_{(1)}\lambda^j{}_{(2)}\lambda^k}. \tag{10}$$

From (4), (9), and (10) it follows that

$$k_{12} = eK_1K_2 + \frac{\overline{R}_{\alpha\beta\gamma\delta}{}_{(1)}\xi^\alpha{}_{(2)}\xi^\beta{}_{(1)}\xi^\gamma{}_{(2)}\xi^\delta}{g_{11}g_{22} - g_{12}^2},$$

where ${}_{(p)}\xi^\alpha = {}_{(p)}\lambda^h y^\alpha_{,h}$ is the component in the y's of ${}_{(p)}\lambda^h$. Since, from (3) and (9), $g_{11}g_{22} - g_{12}^2 = (a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma})_{(1)}\xi^\alpha_{(2)}\xi^\beta_{(1)}\xi^\gamma_{(2)}\xi^\delta$, the last equation is equivalent to

$$k_{12} - \bar{k}_{12} = eK_1K_2, \tag{11}$$

where $\bar{k}_{pq}$ is the Riemannian curvature of V_{n+1} for the orientation determined by ${}_{(p)}\xi^\alpha$ and ${}_{(q)}\xi^\alpha$. Since all the quantities in (11) are invariants, this proves the theorem of Gauss for any V_{n+1} and is valid at all points of V_n for which $x \subset \mathfrak{N}$.

Let V_n be a hypersurface of a Riemann space V_{n+1}. Then the difference of the Riemannian curvatures of the V_n and the V_{n+1} at a point for the orientation determined by the directions of two real lines of curvature of V_n at the point, neither of which is tangent to a null vector, is numerically equal to the product of the corresponding normal curvatures; the sign is given by (5) *and thus is determined by the character of the normal to V_n in V_{n+1}.*

4. **The converse of Gauss' theorem for a flat V_{n+1}.** If V_{n+1} is a flat space, $\bar{k}_{pq} \equiv 0$ and

$$k_{pq} = eK_pK_q \tag{12}$$

for every hypersurface for which the elementary divisors of (8) are simple. This is the theorem of Gauss for a flat V_{n+1}. To prove the converse of this theorem, we first prove the lemma:

Given any Riemann space V_{n+1}, an arbitrary point P of this space, and a set of n mutually orthogonal vectors ${}_{(1)}\xi^\alpha, {}_{(2)}\xi^\alpha, \cdots, {}_{(n)}\xi^\alpha$ at P; then there exists a hypersurface V_n in V_{n+1} such that V_n contains the point P and such that the lines of curvature of V_n have the directions ${}_{(1)}\xi^\alpha, {}_{(2)}\xi^\alpha, \cdots, {}_{(n)}\xi^\alpha$ at P.

To prove this lemma we construct the V_n. Let the coordinates y^α of (1) be normal coordinates with the given point P as center so

that the given ennuple ${}_{(p)}\xi^\alpha$ and the vector ${}_{(n+1)}\xi^\alpha$ normal to ${}_{(p)}\xi^\alpha$ at P are the coordinate directions. Then at P,

$$
\begin{aligned}
&a_{\alpha\alpha} = e_\alpha, \qquad a_{\alpha\beta} = 0, \qquad {}_{(\alpha)}\xi^\alpha = 1, \\
&{}_{(\beta)}\xi^\alpha = 0; \qquad [\nu\mu, \gamma]_\alpha = 0, \qquad \frac{\partial a_{\nu\mu}}{\partial y^\gamma} = 0, \qquad \alpha \neq \beta,
\end{aligned} \tag{13}
$$

where $e_\alpha = a_{\beta\gamma} \cdot {}_{(\alpha)}\xi^\beta {}_{(\alpha)}\xi^\gamma$.

We define a hypersurface S by the equations

$$
\begin{aligned}
y^i &= x^i \\
y^{n+1} &= \tfrac{1}{2} A_i x^{i^2},
\end{aligned} \tag{14}
$$

where the A_i are non-zero constants. It is clear that ${}_{(n+1)}\xi^\alpha$ is normal to S. From (3) and (13) it follows that at P

$$
g_{ii} = e_i, \qquad g_{ij} = 0, \qquad i \neq j. \tag{15}
$$

Upon differentiating (3) with respect to x^k, we have

$$
\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial a_{\alpha\beta}}{\partial y^\gamma} y^{\gamma}_{,k} y^{\alpha}_{,i} y^{\beta}_{,j} + a_{\alpha\beta} \frac{\partial^2 y^\alpha}{\partial x^i \partial x^k} y^{\beta}_{,j} + a_{\alpha\beta} y^{\alpha}_{,i} \frac{\partial^2 y^\beta}{\partial x^j \partial x^k}.
$$

It follows from this equation, (13), and (14) that, for $x^i = 0$,

$$
\frac{\partial g_{ij}}{\partial x^k} = 0. \tag{16}
$$

From (6), (13), (14), and (16), we have at P

$$
\Omega_{ii} = e_{n+1} A_i, \qquad \Omega_{ij} = 0, \qquad i \neq j. \tag{17}
$$

Hence from (15) and (17) it follows that S is a V_n which has the given vectors ${}_{(p)}\xi^\alpha$ as the directions of its lines of curvature at the given point. This proves the lemma.

From this result, the converse of Gauss' theorem for a flat V_{n+1}, stated in the introduction, follows immediately. For if (12) holds for every hypersurface in V_{n+1}, the lemma shows that $\bar{k}_{pq}$ must be zero for an arbitrary orientation, hence V_{n+1} is flat.

It is clear that a similar argument may be employed to show that if

$$
k_{pq} - k_0 = eK_p K_q, \qquad k_0 \text{ a constant},
$$

for every hypersurface for which the elementary divisors of (8) are simple, V_{n+1} has constant Riemannian curvature k_0 and conversely.

PRINCETON UNIVERSITY AND
THE INSTITUTE FOR ADVANCED STUDY

MATRIC CONJUGATES IN A RING $R(A)$*

H. S. THURSTON

1. **Introduction.** The concept of conjugate sets of matrices has undergone several modifications since first defined by Taber. Given a matrix M_0, Taber† defined a set $M_1, M_2, \cdots, M_{n-1}$ to be conjugate to M_0 if (a) the M_i are commutative, (b) they have a common characteristic equation, (c) their elementary symmetric functions are scalars and equal to the elementary symmetric functions of the roots of their characteristic equation.

In Taber's paper, the latent roots of M_0 were assumed to be distinct. Franklin‡ generalized the definition so that this restriction is unnecessary. A set $M_1, M_2, \cdots, M_{n-1}$ is conjugate in the sense of Franklin if (a) the M_i are commutative, (b) their elementary symmetric functions are the elementary symmetric functions of the latent roots of M_0.

Further extension of the concept was made by Sokolnikoff.§ Given a matrix M_0 whose minimum equation is $g(x)=\prod_{i=1}^{r}(x-\rho_i)^{\pi_i}=0$, $(\sum \pi_i=m)$, a set $M_1, M_2, \cdots, M_{n-1}$ is conjugate to M_0 with respect to $g(x)=0$, if (a) each M_i is expressible as a polynomial in M_0 with coefficients in the field formed by adjoining the roots ρ_i and the π_ith roots of unity to the field of the elements of M_0, (b) the elementary symmetric functions of the M_i are the elementary symmetric functions of the roots of $g(x)=0$.

Hermann‖ has used the term conjugate in an even broader sense to denote a set of matrices M_i whose elementary symmetric functions are scalars. That is, the M_i are conjugate with respect to any given polynomial $F(x)$ in that their elementary symmetric functions are the elementary symmetric functions of the roots of $F(x)=0$.

In this paper we propose, by use of the principal idempotent elements of a matrix A, to obtain conjugates of each of the above types corresponding to a restricted class of matrices, namely, any given matrix in the ring $R(A)$, where A is any matrix with simple latent roots. The symbolism employed enables us to write down immedi-

* Presented to the Society, December 28, 1937.

† H. Taber, American Journal of Mathematics, vol. 13 (1891), pp. 157–172.

‡ P. Franklin, Annals of Mathematics, (2), vol. 23 (1921), pp. 97–100.

§ E. S. Sokolnikoff, American Journal of Mathematics, vol. 35 (1933), pp. 167–180.

‖ A. Hermann, Compositio Mathematica, vol. 1 (1934), pp. 284–302.

ately a complete set of conjugates of any of these types, thus avoiding much of the laborious effort involved in former methods. In §6, a set is obtained conjugate with respect to the minimum equation of M_0, but more general than the Sokolnikoff type in that the M_i are not expressible as polynomials in M_0.*

2. **Principal idempotent elements.** Let the minimum equation of a matrix A of order n be

$$\psi(\lambda) \equiv \prod_{i=1}^{\mu} (\lambda - \alpha_i)^{\nu_i} = 0, \qquad \sum \nu_i = \nu,$$

where the α_i are distinct. If $\psi(\lambda) \equiv (\lambda - \alpha_i)^{\nu_i}\pi_i(\lambda)$, then the μ functions $\pi_i(\lambda)$ are relatively prime to each other, and there exist μ functions $\rho_i(\lambda)$ such that $\sum_{i=1}^{\mu}\rho_i(\lambda)\pi_i(\lambda) \equiv 1$. The μ functions $e_i(A) = \rho_i(A)\pi_i(A)$, hereafter denoted merely by e_i, are known as the *principal idempotent elements* of A corresponding to α_i. They are linearly independent, each is different from zero, and they satisfy the relations

$$\sum e_i = 1, \qquad e_i^2 = e_i, \qquad e_ie_j = 0, \qquad i \neq j.$$

In this paper we shall assume that A has simple latent roots only, so that $\mu = n$, and the ρ_i are constants. We shall use the symbol $(k_1, k_2, \cdots, k_n)$ to denote the matrix $\sum_{i=1}^{n} k_ie_i$.

3. **Conjugates in the sense of Taber and of Franklin.** It is well known† that the characteristic equation of the matrix $(k_1, k_2, \cdots, k_n)$ is $\prod_{i=1}^{n}(\lambda - k_i) = 0$. If, then, we cyclicly permute the β_i in the matrix $M_0 = (\beta_1, \beta_2, \cdots, \beta_n)$, (where the β_i are not necessarily distinct), we obtain a set

$$M_i = (\beta_{i+1}, \beta_{i+2}, \cdots, \beta_i), \qquad i = 1, 2, \cdots, n-1,$$

having a common characteristic equation $\prod_{j=1}^{n}(\lambda - \beta_j) = 0$. Since the M_i also satisfy the other conditions imposed by Taber, they constitute a conjugate set in the sense of Taber. The set is more general than that exhibited by Taber in that the latent roots of M_0 are not necessarily distinct.

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* In all of these types it is implied that the matrices M_i are all of a specified order n. A. R. Richardson, in a recent paper (Quarterly Journal of Mathematics, Oxford Series, vol. 7 (1936), pp. 256–270), has exhibited a set of n matrices each of order $n!$ which (a) satisfy the same minimum equation $f_n(x) = 0$, (b) are commutative, and (c) whose elementary symmetric functions are equal to the elementary symmetric functions of the roots of $f_n(x) = 0$.

† Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, New York, 1934, p. 26.

To obtain a set conjugate in the sense of Franklin, we write the symbols for $M_0, M_1, M_2, \cdots, M_{n-1}$ in a square array so that each β_j, $(j=1, 2, \cdots, n)$, occurs once and only once in each column of the array. This insures that the elementary symmetric functions of the M_i shall be the elementary symmetric functions of the β_j, the latter being the roots of the characteristic equation of M_0.

4. **Hermann's conjugates.** Let $F(x)$ be any polynomial with scalar coefficients, having distinct zeros $\beta_1, \beta_2, \cdots, \beta_t$. If we choose any n of these, $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n}$ (not necessarily distinct), the matrix $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ is a root of $F(x)=0$; the matrix $M_i=(\beta_{i_1+i}, \beta_{i_2+i}, \cdots, \beta_{i_n+i})$, i_k+i being reduced modulo t, is also a root of $F(x)=0$, and is called by Hermann the ith *conjugate* of M_0. The matrices M_i are commutative, and their elementary symmetric functions are equal to those of the roots of $F(x)=0$.

5. **Conjugates in the sense of Sokolnikoff.** In order to obtain conjugates of this type, two theorems are necessary.

THEOREM 1. *If among the β's in $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ there are m which are distinct, say $\beta_1, \beta_2, \cdots, \beta_m$, then the minimum equation of M_0 is $\prod_{j=1}^{m}(\lambda-\beta_j)=0$.**

If we now let the polynomial $F(x)$ of the preceding section be $\prod_{j=1}^{m}(\lambda-\beta_j)$ and form the conjugates M_i, $(i=1, 2, \cdots, m-1)$, according to the rule indicated in that section, we can prove the following theorem:

THEOREM 2. *Each M_i can be expressed as a polynomial in M_0, with coefficients in the field formed by adjoining the β_j to the field of the elements of M_0.*

We have $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ and $M_i=(\beta_{i_1+i}, \beta_{i_2+i}, \cdots, \beta_{i_n+i})$, $(i=1, 2, \cdots, m-1)$. Assume that each M_i can be expressed in the form $M_i=\sum_{k=1}^{m} a_k M_0^{m-k}$. Since the e_i are linearly independent, in determining the a_k we are led to a system of m non-homogeneous equations whose determinant is

$$\pm \Delta = \begin{vmatrix} \beta_1^{m-1} & \beta_1^{m-2} & \cdots & 1 \\ \beta_2^{m-1} & \beta_2^{m-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_m^{m-1} & \beta_m^{m-2} & \cdots & 1 \end{vmatrix} = \prod_{i<j} (\beta_i - \beta_j).$$

* Wedderburn, loc. cit

The β_j being distinct, Δ is not zero, and the a_k may be uniquely determined.

Thus the set $M_0, M_1, \cdots, M_{m-1}$ may be considered a very elementary set in the sense of Sokolnikoff. In fact, this is the special case referred to by Sokolnikoff in which the roots of the minimum equation are distinct. It can be readily shown that if $M_1=\Theta(M_0)$, then $M_i=\Theta^{i-1}(M_0)$, $(i=1, 2, \cdots, m-1)$, and $M_0=\Theta^m(M_0)$.*

6. **Other conjugate sets.** If we merely permute the β's in $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ to form the m matrices $M_0, M_1, \cdots, M_{m-1}$, in such a way that distinct β's shall occupy the jth place in each M_i, the set obtained is conjugate relative to the minimum equation of M_0, but, in general, it is impossible to express the other matrices as polynomials in M_0. For example, if $M_0=(\beta_1, \beta_1, \beta_2, \beta_3, \beta_1)$ we may choose $M_1=(\beta_3, \beta_3, \beta_3, \beta_2, \beta_3)$ and $M_2=(\beta_2, \beta_2, \beta_1, \beta_1, \beta_2)$. The set is conjugate relative to $\prod_{j=1}^{3}(\lambda-\beta_j)=0$, but any attempt to express M_1 or M_2 as polynomials in M_0 leads to a system of inconsistent equations. It will be observed that these matrices do not, in general, have a common minimum equation, a property which was possessed by the set obtained in the preceding section.

Finally, we note that this set and that of the Franklin type are not uniquely determined. In the other types described, the law by which M_i was obtained determines the latter uniquely.†

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* Compare Sokolnikoff, loc. cit., corollary, p. 175.

† This is not meant to imply that only one set exists corresponding to M_0, but rather that the law, *as stated*, yields a unique set. Other sets may be determined by a modification of the law. For example, if $M_0=(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3)$, we have the sets

$$\{(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3), (\beta_2, \beta_1, \beta_3, \beta_3, \beta_4), (\beta_3, \beta_2, \beta_4, \beta_4, \beta_1), (\beta_4, \beta_3, \beta_1, \beta_1, \beta_2)\};$$
$$\{(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3), (\beta_2, \beta_3, \beta_4, \beta_4, \beta_1), (\beta_3, \beta_2, \beta_1, \beta_1, \beta_4), (\beta_4, \beta_1, \beta_3, \beta_3, \beta_2)\};$$
$$\{(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3), (\beta_2, \beta_3, \beta_1, \beta_1, \beta_4), (\beta_3, \beta_1, \beta_4, \beta_4, \beta_2), (\beta_4, \beta_2, \beta_3, \beta_3, \beta_1)\},$$

all three of which satisfy the requirements of a conjugate set in the sense of Sokolnikoff.

ABSTRACT RESIDUATION OVER LATTICES*

R. P. DILWORTH

Introduction. The idea of residuation goes back to Dedekind [3],† who introduced it in the theory of modules. It has since had extensive applications in the theory of algebraic modular systems [6], in the theory of ideals [8], and in certain topics of arithmetic [9]. On account of its fundamental role in several fields of modern algebra, it is desirable to consider residuation abstractly. A postulational treatment also is a necessary preliminary to the investigation of the structure properties of the residual. We give such an abstract formulation.

In a commutative ring with unit element the residual of an ideal B with respect to an ideal A, written $A:B$, is an ideal with the properties $A \supset (A:B)B$; if $A \supset XB$, then $A:B \supset X$. Although the residual is defined in terms of multiplication, most of its important properties are concerned with the cross-cut and union of ideals. Hence we shall consider a residual defined over a system having only these two operations, that is, over a *lattice* [2]. As an example of a system having a residual but no ordinary multiplication we consider in §5 residuation in a Boolean algebra.

In §1 the postulates for abstract residuation are given. Equality is taken as an undefined relation with cross-cut, union, and residual as undefined connections. In §2 we list a few systems satisfying the postulates. In §3 it is shown that the system defined by the postulates is a lattice and that the residual has all of its important properties which are independent of multiplication. Consistency and independence proofs are given in §4.

I wish to express my thanks to Professors Morgan Ward and E. T. Bell for their many suggestions and helpful criticisms during the preparation of this paper.

1. **Postulates for residuation.** Let Σ be a set of elements A, B, C, $\cdots$; and let $=$, $[\ ,\ ]$, $(\ ,\ )$, and $:$ be relations, satisfying the postulates i–iv; 1–3; I–V. In what follows, $\circ$ denotes an arbitrary one of the relations $[\ ,\ ]$, $(\ ,\ )$, $:$ and the letters A, B, C, $\cdots$, appearing in the statement of the postulates indicate arbitrary elements of Σ.

POSTULATE i. *$A \circ B$ is in Σ whenever A and B are in Σ.*

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* Presented to the Society, November 27, 1937.

† Numbers in square brackets refer to the bibliography at the end of the paper.

POSTULATE ii. *If* $A=B$, *then* $C \circ A = C \circ B$ *and* $A \circ C = B \circ C$.*
POSTULATE iii. *If* $A=B$ *and* $B=C$, *then* $A=C$.
POSTULATE iv. *If* $A=B$, *then* $B=A$.
POSTULATE 1. $[[A, B], C] = [B, [A, C]]$.†
POSTULATE 2. $[A, A] = A$.
POSTULATE 3. *There is an element* I *in* Σ *such that* $[A, I] = A$ *for all* A *in* Σ.

As immediate deductions from these postulates we have:

1.1. $[I, A] = [A, I] = A$.

PROOF. $[I, A] = [I, [A, I]] = [[A, I], I] = [A, I] = A$ by ii, 1, 3, iii.

1.2. *The element* I *in Postulate* 3 *is unique.*

POSTULATE I. $A:A = I$.
POSTULATE II. $(A:B):C = (A:C):B$.
POSTULATE III. $A:(B, C) = [A:B, A:C]$.
POSTULATE IV. $[A, B]:C = [A:C, B:C]$.
POSTULATE V. *If* $A:B = B:A = I$, *then* $A = B$.

DEFINITION 1. $A:B = I$ is written $A \supset B$.
DEFINITION 2. $[A, B] = B$ is written $A > B$.

2. **Examples.** We list a few systems satisfying the postulates i–V.

1. Let Σ be the set of ideals in a commutative ring with unit element. Let $[\ ,\ ]$ and $(\ ,\ )$ be the cross-cut and union respectively. Let $A:B$ be defined by $A \supset (A:B)B$, if $A \supset XB$, $A:B \supset X$.

2. Let Σ be the set of positive integers with $[\ ,\ ]$ and $(\ ,\ )$ the L. C. M. and G. C. D. respectively. Let $A:B$ be defined by $A/(A, B)$ with $I=1$.

3. As in 2, let Σ be the set of positive integers with $[\ ,\ ]$ and $(\ ,\ )$ defined as max $(\ ,\ )$ and min $(\ ,\ )$ respectively. If now $A:B$ is defined by max $(0, A-B)$ and 0 is taken to be the element I, the postulates are satisfied.

4. Let Σ be the integers $\leqq n$ with $[\ ,\ ]$ and $(\ ,\ )$ defined by min $(\ ,\ )$ and max $(\ ,\ )$, respectively. Define $A:B$ as min $(n, n+A-B)$ with $I=n$.

5. Let Σ be a Boolean algebra with $[\ ,\ ]$, $(\ ,\ )$ the Boolean operations $'$, $\vee$ respectively. Let $A:B = A \vee B'$.

* It is understood that the relations in the postulates hold whenever the respective elements and the indicated combinations are in Σ.

† To facilitate obtaining an independence example, the commutative and associative laws have been combined in one postulate.

3. **Deductions from the postulates.** † We have, first,

2.1. $A = A$ by 3, iv.

2.12. *$A \supset B$ and $B \supset A$ is equivalent to $A = B$*‡ by V, Definition 1.

2.13. *If $M:A = M:B$ for all M in Σ, then $A = B$.*

PROOF. $M = A$, B respectively give

2.14. $A:B = A:A = I$ and $B:A = B:B = I$ by ii, I, iv.

Hence $A = B$ by V.

2.15. $[A, B] = [B, A]$.

PROOF. $[A, B] = [[A, B], I] = [B, [A, I]] = [B, A]$ by 3, 1, 3, ii.

2.16. $(A, B) = (B, A)$ by III, 2.15, 2.13.

2.17. $[[A, B], C] = [A, [B, C]]$ by 2.15, 1.

2.18. $((A, B), C) = (A, (B, C))$ by III, 2.17, III, 2.13.

From 2.17 and 2.18 we may write $[A, B, C]$ for $[[A, B], C]$ and (A, B, C) for $((A, B), C)$. Generally $[A_1, A_2, \cdots, A_n]$, $(A_1, A_2, \cdots, A_n)$ are unambiguous.

*2.19. $M:(A_1, A_2, \cdots, A_n) = [M:A_1, M:A_2, \cdots, M:A_n]$ by induction from III.

*2.2. $[A_1, A_2, \cdots, A_n]:M = [A_1:M, A_2:M, \cdots, A_n:M]$ by induction from IV.

2.22. $(A, A) = A$.

PROOF. $M:(A, A) = [M:A, M:A] = [M, M]:A = M:A$ by III, IV, 2. Hence $(A, A) = A$ by 2.13.

2.23. $(A, B) \supset A$.

PROOF. $(A, B):A = [(A, B):A, I] = [(A, B):A, (A, B):(A, B)] = (A, B):(A, (A, B)) = (A, B):((A, A), B) = (A, B):(A, B) = I$ by 3, I, III, 2.18, 2.22, I.

2.24. *If $A > B$, then $A \supset B$.*

PROOF. $[A, B] = B$ gives $A:B = [A:B, I] = [A:B, B:B] = [A, B]:B = B:B = I$ by 3, I, IV, I.

2.25. *If $[A, B] = B$, then $(A, B) = A$.*

PROOF. $A:(A, B) = [A:A, A:B] = [I, A:B] = A:B = I$ by III, I, 3, 2.24. Hence $(A, B) = A$ by 2.23, V.

2.26. $(A, I) = (I, A) = I$ by 2.25, 3, 2.16.

† The theorems giving the essential properties of the residual will be starred.

‡ By "equivalent" we mean formal equivalence.

*2.27. $I:A=I$.

PROOF. $I:A=[I,\ I:A]=[I:I,\ I:A]=I:(I,\ A)=I:I=I$ by 1.1, I, III, 2.26, I.

*2.28. $A:I=A$.

PROOF. $(A:(A:I)):I=(A:I):(A:I)=I$ by II, I. $I:(A:(A:I))=I$ by 2.27. Hence $A:(A:I)=I$ by V. But $(A:I):A=(A:A):I=I:I=I$ by II, I. Hence $A:I=A$ by V.

2.29. $A \supset [A, B]$ by 2.17, 2, 2.24.

2.3. *If* $A \supset B$, *then* $A>B$.

PROOF. $[A,\ B]:B=[A:B,\ B:B]=[I,\ I]=I$ by IV, I, 2. But $B:[A,\ B]=I$ by 2.29, 2.15, Definition 1. Hence $[A,\ B]=B$ by V.

2.31. $A \supset B$ *is equivalent to* $A>B$ by 2.24, 2.3.

2.32. $(A,\ [A,\ B])=A$ by 2.29, 2.31, 2.25.

2.33. $[A,\ (A,\ B)]=A$ by 2.23, 2.31, Definition 2.

2.34. Σ *is a lattice.*

PROOF. We show that Birkhoff's axioms L1–L4† are satisfied if we take $[\ ,\ ]\equiv \cap$. For L1 is i; L2 is 2.15 and 2.16; L3 is 2.17 and 2.18; L4 is 2.32 and 2.33.

*2.35. $A:(A:B) \supset B$ by II, I.

*2.4. *If* $B \supset C$, *then* $A:C \supset A:B$.

PROOF. $A:(A:B) \supset B$ and $B \supset C$ by 2.35. Hence $(A:(A:B)):C = I$ by 2.34, Definition 1. Then $(A:C):(A:B) = (A:(A:B)):C = I$ by II.

*2.41. $A:B=A:(A,\ B)$ by III, I, 1.1.

*2.42. $[A,\ B]:B=A:B$ by II, I, 3.

*2.43. $I \supset A:B \supset A$ by 2.27, II, I, 2.27.

*2.44. $A:(A:(A:B))=A:B$.

PROOF. $[A:(A:(A:B))]:(A:B)=[A:(A:B)]:[A:(A:B)]=I$ by II, I. $(A:B):[A:(A:(A:B))]=[A:[A:(A:(A:B))]]:B=I$ by II. Since $A:[A:(A:(A:B))] \supset A:(A:B)$ and $A:(A:B) \supset B$ by 2.35, hence $A:(A:(A:B))=A:B$ by V.

It will be noted from 2.44 that $A:(A:B)$ and $A:B$ are mutually residual with respect to A.

*2.45. *If* $A:C=A:B$, *then* $A:(A:B) \supset C$ by II, I.

† G. Birkhoff, *On the lattice theory of ideals*, this Bulletin, vol. 40 (1934), p. 613. L1, L2, L3, and L4 are his axioms for a lattice.

These theorems are sufficient to show that the usual properties of the residual are deducible from the postulates of §1.

Since the residual as here considered is independent of multiplication, there is a residual completely dual to that defined above. The dual may be defined by the postulates I′–V′:

POSTULATE I′. *$A:A=E$ where E is the null element of the lattice.*
POSTULATE II′. $(A:B):C=(A:C):B$.
POSTULATE III′. $A:[B, C]=(A:B, A:C)$.
POSTULATE IV′. $(A, B):C=(A:C, B:C)$.
POSTULATE V′. *If $A:B=B:A=E$, then $A=B$.*

Thus, for the integers 0, 1, 2, $\cdots$, n as the given set, let $[\ ,\ ]$, $(\ ,\ )$ be defined as min $\langle\ ,\ \rangle$, max $(\ ,\ )$ respectively with $E=0$ and $I=n$. Then $A:B\equiv\min(n, n+A-B)$ is a residual satisfying the postulates of §1, while $A:B\equiv\max(0, A-B)$ is a residual satisfying the second set of postulates.

4. Consistency and independence proofs. †

	$[\ ,\ ]$	$(\ ,\ )$	$:$	
Consistency	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 1 \end{array}$	
Independence				
1	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{array}$	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & 3 & 3 \end{array}$	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 2 & 1 \end{array}$	$[[1, 3], 2]\neq[3, [1, 2]]$
2	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 1 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 1 \end{array}$	$[2, 2]\neq 2$
3	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{array}$	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 3 & 3 \\ 2 & 3 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{array}$	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 2 & 1 \end{array}$	$[2, 1]\neq 2$, $[3, 2]\neq 3$, $[2, 3]\neq 2$
I	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$	$2:2\neq 1$

	[,]	(,)	:	
II	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{array}$	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{array}$	$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 3 & 3 & 2 & 1 \end{array}$	$(3:1):2 \neq (3:2):1$
III	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 1 \end{array}$	$2:(2, 1) \neq [2:2, 2:1]$
IV	$\begin{array}{c\|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{array}$	$\begin{array}{c\|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 \\ 3 & 1 & 2 & 3 & 3 \\ 4 & 1 & 2 & 3 & 4 \end{array}$	$\begin{array}{c\|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 1 & 1 \\ 4 & 4 & 2 & 2 & 1 \end{array}$	$[4, 3]:2 \neq [4:2, 3:2]$
V	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 2 \end{array}$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}$	$2:1 = 1:2 = 1$ but $2 \neq 1$.

† The independence examples for i–iv are omitted.

5. **Residuation in a Boolean algebra** [1]. If we take Σ to be a Boolean algebra and interpret [,], (,) as the Boolean operations $\cdot$, $\vee$ respectively, then it is readily verified that Σ satisfies postulates i–V if we define residuation by $A:B \equiv A \vee B'$. Moreover we have the following theorem:

THEOREM. *Let Σ be a Boolean algebra and let* [,], (,) *be the Boolean operations $\cdot$, $\vee$ respectively. Then the only Boolean operation satisfying postulates* I–V *is $A:B = A \vee B'$.*

PROOF. Write $A:B$ as a general Boolean function of A and B

$$A:B = K_1AB \vee K_2AB' \vee K_3A'B \vee K_4A'B'.$$

Then

$$1:1 = K_1 = 1, \qquad 0:1 = K_3 = 0,$$

$$1:0 = K_2 = 1, \qquad 0:0 = K_4 = 1.$$

Hence

$$A:B = AB \vee AB' \vee A'B' = A(B \vee B') \vee A'B' = A \vee A'B'$$

$$= (A \vee AB') \vee A'B' = A \vee (A \vee A')B' = A \vee B',$$

and as above this is sufficient that postulates I–V be satisfied.

With this definition of $A:B$, $A \supset B$ becomes the usual inclusion relation of the algebra of classes [5].

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A NOTE ON THE MAXIMUM PRINCIPLE FOR ELLIPTIC DIFFERENTIAL EQUATIONS

FRITZ JOHN

Let $u(x_1, \cdots, x_n)$ denote a twice continuously differentiable function of $x_1, \cdots, x_n$ in some region R. We write $\partial u/\partial x_i = u_i$, $\partial^2 u/\partial x_i \partial x_k = u_{ik}$, and occasionally (x) for $(x_1, \cdots, x_n)$. A point $(c) = (c_1, \cdots, c_n)$ of R may be called a *proper* maximum of u, if

$$u_i(c) = 0 \qquad \text{for} \qquad i = 1, \cdots, n,$$

$$\sum_{i,k} u_{ik}(c)\xi_i\xi_k < 0 \qquad \text{for all} \qquad (\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0).$$

A partial differential equation

$$\sum_{i,k} a_{ik}(x)u_{ik}(x) + \sum_i b_i(x)u_i(x) = 0 \tag{1}$$

(where the a_{ik} and b_i are defined in R) is called *elliptic* if for every (x) of R

$$\sum_{i,k} a_{ik}(x)\xi_i\xi_k \geqq 0$$

for all $(\xi_1, \cdots, \xi_n)$ and <0 for some $(\xi_1, \cdots, \xi_n)$.

It is well known, that a solution u of (1) can not have a proper maximum.* For if u had a proper maximum at $(c_1, \cdots, c_n)$, then $\sum_{i,k} a_{ik}(c)u_{ik}(c)=0$. If A and U denote respectively the matrices $(a_{ik}(c))$ and $(u_{ik}(c))$, this may be written: Trace $(A \cdot U)=0$. By a suitable orthogonal transformation A may be transformed into a diagonal matrix $A'=(a_i'\delta_{ik})$, U going over into $U'=(u'_{ik})$ by the same transformation. As the trace of $A \cdot U$ is preserved, we have $\sum_i a_i' u'_{ii}=0$; on the other hand, as A' still belongs to a semi-definite quadratic form and U' to a negative definite one, we have $a_i' \geqq 0$ for $i=1, \cdots, n$, but <0 for some i, and $u'_{ii}<0$ for all i. This leads to a contradiction.

A second important property of the solutions u of (1) is that they form a module, that is, that every linear combination with constant coefficients is again a solution.

We shall prove that these two properties are also sufficient to characterize a family of functions as solutions of an elliptic equation (1).

THEOREM. *Let F be any family of twice continuously differentiable functions $u(x_1, \cdots, x_n)$ defined in R, such that*

(a) *the functions of F form a module,*

(b) *no function of F has a proper maximum.*

Then there is an elliptic differential equation (1) *satisfied by all functions of F.*

PROOF. Let $(c)=(c_1, \cdots, c_n)$ be a point of R. Let ϕ be the submodule of functions u of F for which $u_i(c)=0$ for $i=1, \cdots, n$. Let $Q(\xi_1, \cdots, \xi_n)$ denote the quadratic form $\sum_{i,k} u_{ik}(c)\xi_i\xi_k$ for any u in ϕ; Q is certainly not negative definite. These quadratic forms form again a module M. Let $Q_1, Q_2, \cdots, Q_m$ form a basis of this module $(m \leqq n(n+1)/2)$, such that for every Q of M

$$Q(\xi_1, \cdots, \xi_n) = \sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \cdots, \xi_n)$$

with certain constants λ_i. We know that for every $(\lambda_1, \cdots, \lambda_m)$ there are $(\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0)$ such that

$$\sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \cdots, \xi_n) \geqq 0. \tag{2}$$

From this we can easily conclude that the Q_i satisfy a linear relation

* Cf. Encyklopädie der Mathematischen Wissenschaften, vol. 2, 1.1, p. 522, or Picard, *Traité d'Analyse*, 3d ed., vol. 2, p. 29 for the case $n=2$. The subsequent proof follows Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. 2.

with positive coefficients.* For let Σ denote the set of points with coordinates $(Q_1(\xi), Q_2(\xi), \cdots, Q_m(\xi))$ in an m-dimensional space, where $(\xi)=(\xi_1, \cdots, \xi_n)$ varies over all points of the unit sphere $\xi_1^2+\xi_2^2+\cdots+\xi_n^2=1$. The set Σ is closed and finite. The relation (2) may be interpreted as stating that every half-space bounded by a plane through the origin contains points of Σ. Thus the origin is contained in the convex extension of Σ. Then there exists a finite set of points $P_1, \cdots, P_r$ of Σ and positive numbers $\mu_1, \cdots, \mu_r$, such that the origin is the center of mass of the masses μ_i placed at the vertices P_i.† Let $(\xi_1^i, \cdots, \xi_n^i)$ be the point $(\xi_1, \cdots, \xi_n)$ corresponding to P_i. Then

$$\sum_{j=1}^{r} \mu_j Q_k(\xi_1^j, \cdots, \xi_n^j) = 0$$

for $k=1, \cdots, m$, and consequently

$$\sum_{j=1}^{r} \mu_j Q(\xi_1^j, \cdots, \xi_n^j) = 0$$

for every Q in M. Thus

$$\sum_{j=1}^{r} \sum_{i,k} \mu_j u_{ik}(c_1, \cdots, c_n)\xi_i^j \xi_k^j = 0$$

for u in ϕ. Putting $\sum_j \mu_j \xi_i^j \xi_k^j = a_{ik}(c)$, we have

$$\sum_{i,k} a_{ik}(c)u_{ik}(c) = 0$$

for u in ϕ. Besides

$$\sum_{i,k} a_{ik}(c)\xi_i\xi_k = \sum_{i,j} \mu_j(\xi_i^j \xi_i)^2 \geqq 0$$

and >0 for some $(\xi_1, \cdots, \xi_n)$.

Now let $u(x_1, \cdots, x_n)$ denote an arbitrary function of F. The vectors $(y_1, \cdots, y_n)=(u_1(c), u_2(c), \cdots, u_n(c))$ again form a module N, if u varies over F for fixed (c). Let $(y_1^1, y_2^1, \cdots, y_n^1)$, $(y_1^2, \cdots, y_n^2)$, $\cdots$, $(y_1^s, \cdots, y_n^s)$ form a basis of N, $(s \leqq n)$. Without restriction of generality we may assume that this basis forms an orthogonal system:

$$\sum_{l=1}^{n} y_l^i y_l^k = \delta_{ik}, \qquad i, k = 1, \cdots, s.$$

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* Cf. L. L. Dines, this Bulletin, vol. 42, p. 357, and the paper of R. W. Stokes Transactions of this Society, vol. 33, p. 782 et seq.

† Bonnesen-Fenchel, *Theorie der Konvexen Körper*, p. 9.

Let $u^1(x, c), u^2(x, c), \cdots, u^s(x, c)$ be the functions of F corresponding to the vectors

$$u_i^k(c, c) = y_i^k .$$

Then for every u in F

$$u_i(c) = \sum_{j=1}^{s} \lambda_j y_i^j$$

with

$$\lambda_j = \sum_{k=1}^{n} u_k(c) y_k^j .$$

Thus

$$u_i(c) = \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u_i^j(c, c).$$

Consider the function

$$\bar{u}(x) = u(x) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u^j(x, c).$$

Then $\bar{u}$ is in F. We have

$$\bar{u}_i(c) = u_i(c) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u_i^j(c, c) = 0.$$

Hence $\bar{u}$ is in ϕ. Consequently

$$0 = \sum_{i,k} a_{ik}(c) \bar{u}_{ik}(c) = \sum_{i,k} a_{ik}(c) u_{ik}(c) - \sum_{i,j,k,h} a_{ik}(c) u_k(c) u_k^j(c, c) u_{ih}^j(c, c).$$

Thus $u(x)$ satisfies the elliptic equation

$$0 = \sum_{i,k} a_{ik}(x) u_{ik}(x) - \sum_{i,j,k,h} a_{ik}(x) u_k^j(x, x) u_{ih}^j(x, x) u_k(x).$$

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BIRATIONAL TRANSFORMATIONS IN 4-SPACE AND 5-SPACE

A. R. WILLIAMS

The purpose of this paper is to describe certain Cremona transformations in 4-space and 5-space, especially the former. The transformations in question are interesting on account of analogies with transformations in ordinary space, because they can be simply expressed and effectively studied by the use of equations, as well as synthetically, and because of the classic nature of the loci involved. A rational quartic curve in S_4 and a Veronese surface in S_5 are of special importance.

1. **The T_{2-4} in S_4.** Through C_1^4, a rational quartic curve in S_4, pass ∞^5 hyperquadrics. Three of these intersect in a residual curve of order 4, also rational, that meets C_1^4 six times. Therefore through the C_1^4 and a general point O of S_4 pass ∞^4 hyperquadrics, four of which have one variable point in common. Thus the ∞^4 hyperquadrics, which may be conveniently called the system (ϕ), may be put in 1:1 correspondence with the hyperplanes of the same or another S_4. The C_1^4 and the point O constitute the fundamental points of the first space, or the first fundamental system. Three ϕ's have in common a residual rational quartic c to which corresponds a line in the second space. To intersections of c with a hyperplane correspond intersections of the line with a hypersurface of the system (ϕ'). The latter are of order 4, and the transformation is described by the symbol T_{2-4}.

Let the rational quartic C_1^4 be given by $x_0:x_1:x_2:x_3:x_4=t^4:t^3:t^2:t:1$. The determinants of the matrix

$$\left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{matrix} \right\|$$

equated to zero give six linearly independent hyperquadrics containing the curve. Through a general point O $(y_0:y_1:y_2:y_3:y_4)$ of S_4 pass ∞^4 hyperquadrics that contain the curve. One of them is a cone with vertex at O. Its equation is the sum of the six products formed from the above matrix $(x_0x_2 - x_1^2)(y_2y_4 - y_3^2) + (x_0x_3 - x_1x_2)(y_2y_3 - y_1y_4) + \cdots = 0$. We may, however, define the transformation thus:

$$(1) \qquad \begin{aligned} x_0' : x_1' : x_2' : x_3' : x_4' = {} & (x_0x_2 - x_1^2) : (x_0x_3 - x_1x_2) : (x_0x_4 - x_1x_3) : \\ & (x_1x_4 - x_2x_3) : (x_2x_4 - x_3^2). \end{aligned}$$

This is to take for O the point (0:0:1:0:0). It is, to be sure, the intersection of the osculating planes to the curve at (0:0:0:0:1) and (1:0:0:0:0). But that does not render the transformation any less general. Equations (1) can be solved giving

$$
\begin{aligned}
(2)\quad & x_0 : x_1 : x_2 : x_3 : x_4 = x_2'^2(x_0'x_2' - x_1'^2) - x_0'x_2'(x_0'x_4' - x_1'x_3') : \\
& x_1'x_2'(x_0'x_4' - x_1'x_3') + x_0'x_2'^2x_3' : x_0'x_2'^2x_4' - (x_0'x_4' - x_1'x_3')^2 : \\
& x_2'x_3'(x_0'x_4' - x_1'x_3') + x_1'x_2'^2x_4' : x_2'^2(x_2'x_4' - x_3'^2) \\
& - x_2'x_4'(x_0'x_4' - x_1'x_3').
\end{aligned}
$$

Since the expressions on the right of (2), except the third, contain the factor x_2' it follows that to the point (0:0:1:0:0) corresponds the hyperplane $x_2' = 0$. It appears also from (2) that the quadric surface defined by $x_2' = 0$, $x_0'x_4' - x_1'x_3' = 0$ lies doubly on the quartic homaloids, the system (ϕ'). The jacobian of the expressions on the right of (1) consists of two factors, $x_0x_4 - x_1x_3$ and

$$
\begin{vmatrix}
x_0 & x_1 & x_2 \\
x_1 & x_2 & x_3 \\
x_2 & x_3 & x_4
\end{vmatrix}.
$$

These two equated to zero give the unique hypercone of the system (ϕ) whose vertex is at O(0:0:1:0:0) and the cubic hypersurface which is the locus of chords of C_1^4. They constitute the first principal system, that is, the loci in the first space that correspond to the fundamental points in the second space. It is seen immediately from equations (1) that to any point on the hypercone $x_0x_4 - x_1x_3 = 0$ corresponds a point on the quadric surface $F_2'^2$, that is, $x_2' = 0$, $x_0'x_4' - x_1'x_3' = 0$. To the cubic hypersurface, locus of the chords of C_1^4, corresponds a rational quartic surface $F_2'^4$ whose parametric representation will be obtained shortly. To C_1^4 corresponds a sextic hypersurface in the second space, since C_1^4 is met six times by the residual curve of intersection of any three ϕ's. In fact, the jacobian of the expressions on the right of (2) is of order 15 and consists of the factor $x_2'^3$ and a sextic factor taken twice, the latter corresponding to C_1^4. We can obtain this sextic hypersurface by simply substituting from (2) into (1). We find $x_0' : x_1' : x_2' : x_3' : x_4' = x_0'G' : x_1'G' : x_2'G' : x_3'G' : x_4'G'$, where G' is of degree 7, and consists of x_2' and the sextic factor $x_2'^3(x_0'x_2'x_4' - x_0'x_3'^2 - x_1'^2x_4') - x_2'^2(x_0'x_4' - x_1'x_3')(2x_0'x_4' + x_1'x_3') + (x_0'x_4' - x_1'x_3')^3$. The latter equated to zero gives the sextic hypersurface in question. The quadric surface $F_2'^2$, that is, $x_2' = 0$, $x_0'x_4' - x_1'x_3' = 0$, is evidently a triple surface on it. The sextic hypersurface is composed of ∞^1 planes that correspond to the points

of C_1^4. All points of a chord of C_1^4 in the first space present the same condition to (ϕ), and any ϕ that contains a general point of such a chord contains all of it. Hence to each chord of C_1^4 corresponds a single (fundamental) point in the second space. In fact we note that a general point of the line joining the points whose parameters are t_1 and t_2 has the coordinates $(t_1^4+kt_2^4 : t_1^3+kt_2^3 : t_1^2+kt_2^2 : t_1+kt_2 : 1+k)$. Substituting in (1) we have $x_0' : x_1' : x_2' : x_3' : x_4' = t_1^2t_2^2 : t_1t_2(t_1+t_2) : t_1^2+t_1t_2+t_2^2 : t_1+t_2 : 1$. Setting $t_1t_2=l/n$, $t_1+t_2=m/n$, we have the ratios $l^2 : lm : m^2-ln : mn : n^2$. Hence the surface is rational of order 4. Its hyperplane sections are rational quartic curves whose images in the plane $l:m:n$ are conics. This quartic surface lies simply on the quartic homaloids (ϕ') and doubly on the sextic hypersurface that corresponds to the points of C_1^4. Therefore a quartic homaloid ϕ' meets the sextic hypersurface in four planes which correspond to the points in which the corresponding hyperplane meets C_1^4, and in $F_2'^4$ taken twice and in $F_2'^2$ taken six times.

A general homaloid of the first space has the equation $a_0(x_0x_2-x_1^2) + a_1(x_0x_3-x_1x_2) + a_2(x_0x_4-x_1x_3) + a_3(x_1x_4-x_2x_3) + a_4(x_2x_4-x_3^2)=0$. It contains the chord whose parameters are t_1 and t_2 if $a_0t_1^2t_2^2 + a_1t_1t_2(t_1+t_2)+a_2(t_1^2+t_1t_2+t_2^2)+a_3(t_1+t_2)+a_4=0$. The corresponding hyperplane meets the quartic surface in a rational quartic whose image in the plane $l:m:n$ has the equation $a_0l^2+a_1lm+a_2(m^2-ln) + a_3mn+a_4n^2=0$. The conic in $l:m:n$ may be two intersecting lines or a perfect square. Correspondingly there are ∞^3 hyperplanes that meet the surface in two conics having a common point, and ∞^1 hyperplanes that are tangent to it along a conic. The last system is of order 4 in the sense that 4 such hyperplanes pass through a general point of S_4. Fixing t_1 we see that to the chords projecting C_1^4 from the point t_1 corresponds a conic on the quartic surface $F_2'^4$, which is represented in the plane $l:m:n$ by $l-t_1m+t_1^2n=0$. There are ∞^1 ϕ's which are hypercones having the point t_1 for vertex. To each of them corresponds a hyperplane in the second space that meets $F_2'^4$ in this fixed conic and one variable conic that meets the fixed conic in one point.

The hypercone $x_0x_4-x_1x_3=0$ in the first space deserves further mention. It consists of the ∞^1 trisecant planes of C_1^4 that pass through O (0:0:1:0:0). Thus the members of (ϕ) meet such a plane in the conics of a pencil. To each conic corresponds a point; and to the pencil of conics corresponds a line, since each ϕ in the first space contains one conic of the pencil. As noted above, these lines constitute the quadric surface $F_2'^2$ which is double on the quartic homaloids in the second space, since its points correspond to curves of order 2. Three associated points of C_1^4 whose plane contains (0:0:1:0:0) have

the parameters t, ωt, $\omega^2 t$. The three degenerate conics of the pencil determined by these four points give three points, one of which is $(t^4: -t^3:0: -t:1)$, which lie on the generator of $F_2'^2$ corresponding to the plane as a whole and on the rational quartic curve common to $F_2'^2$ and $F_2'^4$.

2. **Special forms of the transformation.** We may allow O to approach a limiting position on C_1^4, or place O on a chord or tangent to C_1^4, whereupon that chord or tangent becomes part of the curve common to (ϕ). If O approaches a limiting position on C_1^4, the members of (ϕ) acquire a common tangent plane e (not hyperplane) defined by the tangent to C_1^4 and the direction of approach. The ∞^1 hyperplanes through e meet C_1^4 in two associated points P_1 and P_2, and the ∞^1 planes determined by these pairs and the limiting position of O constitute a hypercone, to which corresponds in the manner just described a quadric surface of fundamental points $F_2'^2$. The variable intersection of any three ϕ's passes through O and meets C_1^4 five times. Therefore to O corresponds a hyperplane, and to the general points of C_1^4 correspond planes forming a quintic hypersurface. To the chords of C_1^4 corresponds again a quartic surface of fundamental point $F_2'^4$. If O approaches (0:0:0:0:1) along the line joining the latter to[S] (0:1:1:d:e), the homaloids acquire the common tangent plane $x_0=x_1-x_2=0$; and we have the transformation $x_0' : x_1' : x_2' : x_3' : x_4' = (x_0x_2 - x_1^2) : (x_0x_3 - x_1x_2) : (x_1x_3 - x_2^2) : (x_0x_4 - x_1x_3) : [(x_1x_4 - x_2x_3) - (x_2x_4 - x_3^2)]$. If we form the jacobian of the expressions on the right, the equation of the hypercone corresponding to $F_2'^2$ is seen to be $x_0x_3 - x_0x_2 + x_1^2 - x_1x_2 = 0$. Two associated points on C_1^4 have the parameters t and $t/(t-1)$. That is, the plane determined by them and (0:0:0:0:1) lies in the hypercone just mentioned, and the latter consists of such planes. Two chords joining (0:0:0:0:1) to a pair of associated points give the same point on $x_0' = x_1' = x_2' = 0$, which is, therefore, a double line on the quartic surface $F_2'^4$ and a generator of the second regulus of $F_2'^2$. To the ∞^1 chords joining pairs of associated points corresponds a conic lying in the plane $x_0' - x_1' = 0$, $x_1' - x_2' - x_3' = 0$. This conic and the line $x_0' = x_1' = x_2' = 0$ constitute the intersection of $F_2'^2$ and $F_2'^4$. Solving the equations defining the transformation for $x_0:x_1:x_2:x_3:x_4$, we find that to (0:0:0:0:1) corresponds the hyperplane $x_0' - x_1' = 0$; and the quadric $F_2'^2$ is found to be $x_0' - x_1' = 0$, $x_0'x_4' - x_2'x_3' = 0$, and to be double on the quartic homaloids (ϕ').

More interesting is the case when O is on a chord of C_1^4, that is, when the system (ϕ) consists of the ∞^4 hyperquadrics containing C_1^4 and the chord. Then the residual intersection of three ϕ's is a cubic

which meets the chord once and C_1^4 four times. Hence the transformation is a T_{2-3}. The ∞^1 planes determined by the chord and the points of C_1^4 constitute a hyperquadric cone of the second species; that is, every point of the chord is a vertex. Such a plane is cut by the members of (ϕ) in the lines of a pencil with center on C_1^4. To each such line corresponds a point, and to the plane as a whole corresponds a line. To the totality of these lines, that is, to the hyperquadric cone, corresponds a quadric surface $F_2'^2$, which, however, is a simple surface on the cubic homaloids (ϕ') since its points correspond to straight lines. To the chords of C_1^4 corresponds a cubic surface. For two ϕ's have three variable chords of C_1^4 in common. To the fixed chord A_1A_2 corresponds a hyperplane since it is met by the variable intersection of three ϕ's in one point. To the individual points of A_1A_2 correspond planes which must, therefore, form an axial pencil. To the points of C_1^4 correspond planes forming a quartic hypersurface.

To express such a transformation algebraically we have the equations $x_0' : x_1' : x_2' : x_3' : x_4' = (x_0x_2 - x_1^2) : (x_0x_3 - x_1x_2) : (x_1x_3 - x_2^2) : (x_1x_4 - x_2x_3) : (x_2x_4 - x_3^2)$. The expressions on the right equated to zero represent five linearly independent hyperquadrics which contain C_1^4 and the chord $x_1 = x_2 = x_3 = 0$ joining the points (0:0:0:0:1) and (1:0:0:0:0) whose parameters are 0 and ∞. To the chord joining the points of C_1^4 whose parameters are t_1 and t_2 corresponds the point $t_1^2t_2^2 : t_1t_2(t_1+t_2) : t_1t_2 : t_1+t_2 : 1$. Writing as before $t_1t_2 = l/n$, $t_1+t_2 = m/n$, we have $x_0' : x_1' : x_2' : x_3' : x_4' = l^2 : lm : ln : mn : n^2$. The expressions on the right vanish for $l = n = 0$. Therefore the surface corresponding to the chords of C_1^4 is a cubic $F_2'^3$. Corresponding to the point (0:1:0) of the plane $l:m:n$ is the line of the surface $x_0' = x_2' = x_4' = 0$. To the lines $l = 0$ and $n = 0$ of the same plane correspond the lines of the surface $x_0' = x_1' = x_2' = 0$, and $x_2' = x_3' = x_4' = 0$. It will be noted that these three lines constitute the intersection of the cubic surface with the hyperplane $x_2' = 0$, and that the second and third lie on the quadric surface $x_2' = 0$, $x_0'x_4' - x_1'x_3' = 0$. This is the quadric surface $F_2'^2$ that corresponds to the hypercone composed of the planes determined by the chord A_1A_2 and the points of C_1^4. And $x_2' = 0$ is the hyperplane that corresponds to the chord A_1A_2. For let P $(k:0:0:0:1)$ be a general point of that chord. By allowing a point Q to approach P in an arbitrary direction we find easily that the plane corresponding to $(k:0:0:0:1)$ is defined by $x_2' = 0$, $x_0' - kx_4' = 0$. Hence to the chord as a whole corresponds the 3-space $x_2' = 0$ made up of the pencil of planes whose axis is $x_2' = x_0' = x_4' = 0$. To the line joining $(k:0:0:0:1)$ and $(t^4:t^3:t^2:t:1)$, a general point of C_1^4, corresponds the point $(kt:k:0:t^2:t)$. All such points lie on the quadric surface

$F_2'^2$ defined by $x_2'=0, x_0'x_4'-x_1'x_3'=0$. When t is fixed and k varies, we get the line $x_2'=0, x_0'-tx_1'=0, x_3'-tx_4'=0$, which corresponds to the plane joining the chord and the point t. When k is fixed and t varies, we get a conic whose plane is defined by $x_2'=0, x_0'-kx_4'=0$. Putting $t=0$ and $k=0$, we have the pair of intersecting lines $x_0'=x_2'=x_3'=0$ and $x_0'=x_1'=x_2'=0$, which correspond respectively to the plane determined by the chord and the tangent at (0:0:0:0:1) and to the cubic cone projecting C_1^4 from the same point. These lines meet the axis $x_0'=x_2'=x_4'=0$ and determine a plane of the pencil tangent to $F_2'^2$. For the plane determined by the chord and the tangent at (1:0:0:0:0) and for the cone projecting C_1^4 from that point, we have the similar pair $x_1'=x_2'=x_4'=0$ and $x_2'=x_3'=x_4'=0$ which give the other plane of the axial pencil tangent to $F_2'^2$. The quadric surface $F_2'^2$ lies simply, and $F_2'^3$ lies doubly, on the quartic hypersurface composed of planes that correspond to the points of C_1^4. They are both simple surfaces on the cubic homaloids (ϕ'). It should be noted that C_1^4 and the chord A_1A_2 constitute a quintic of genus 1; and it is well known that through a quintic of genus 1 pass ∞^4 hyperquadrics three of which have for residual intersection a cubic that meets the quintic five times. A Cremona transformation can be constructed on that basis.

3. **The T_{2-2} in S_5.** Coming now to 5-space, we see that there is a simple Cremona transformation which is quadratic both ways, and which therefore may be an involution. There are ∞^5 hyperquadrics which have a Veronese surface F_2^4 in common. Three of them have a residual intersection which is also a Veronese surface, and which has a sextic curve of genus 1 in common with F_2^4. A fourth hyperquadric of the system meets the variable Veronese surface in a curve of order 8 consisting of the sextic and a conic c which meets the sextic curve, and so F_2^4, three times. Hence five hyperquadrics of the system have one variable point in common. We have thus a transformation in which the conic c corresponds to a line, and which is therefore a T_{2-2}. The first principal system consists of the ∞^2 planes of the conics on F_2^4. To each such plane corresponds a point. To the ∞^1 such planes passing through a point P of F_2^4 correspond the points of a conic. For a hyperquadric ϕ contains two of them. In fact the tangent hyperplane to ϕ at P meets F_2^4 in a quartic curve which consists of two of the conics lying on F_2^4 and passing through P. The planes of these conics lie on ϕ. Thus each ϕ contains ∞^1 planes of conics belonging to F_2^4, and two ϕ's have four such planes in common. Hence the surface of fundamental points in the second space is of order 4 and contains ∞^2 conics. It is therefore a Veronese surface $F_2'^4$. To the points of F_2^4

correspond the ∞^2 planes of the conics on $F_2'^4$. They constitute a cubic hypersurface, since it is met by a line in three points corresponding to the points in which the residual intersection (conic) of four ϕ's meets F_2^4.

This transformation in its involutory form may be expressed by a very simple system of equations. The Veronese surface F_2^4 is rational, and its ∞^5 hyperplane sections correspond to the conics of the plane of representation. Hence the points of F_2^4 are given by $x_0:x_1:x_2:x_3:x_4:x_5=l^2:lm:ln:m^2:mn:n^2$. Thus the minors of

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix}$$

equated to zero give six linearly independent hyperquadrics containing F_2^4. The determinant itself equated to zero is the equation of the locus of the planes of the ∞^2 conics that lie on F_2^4. These planes contain all the chords of the surface, since through any two points of it passes one of the conics. This suggests the equations of transformation (1) $x_0':x_1':x_2':x_3':x_4':x_5'=(x_3x_5-x_4^2):(x_2x_4-x_1x_5):(x_1x_4-x_2x_3):(x_0x_5-x_2^2):(x_1x_2-x_0x_4):(x_0x_3-x_1^2)$. The jacobian of the expressions on the right is of course the above determinant taken twice. If we substitute on the right for the unprimed letters the same functions of the primed letters we get $x_0':x_1':x_2':x_3':x_4':x_5'=x_0'G':x_1'G':x_2'G':x_3'G':x_4'G':x_5'G'$, where G' is the above determinant in the primed letters, and $G'=0$ gives the locus of points corresponding to the points of F_2^4. Thus we have a simple involution by which the points of S_5 are paired. The points of the Veronese surface F_2^4, however, correspond to the planes of its conics, and vice versa. This correspondence is double in the sense that if a point and plane of F_2^4 correspond to each other by the direct transformation they do also by the reverse. In fact such a point and plane pair may be defined in the following way by the same three numbers. Let $al+bm+cn=0$ be the equation of a line in the plane $l:m:n$ representing the Veronese surface. Three points on this line are $(0:c:-b)$, $(c:0:-a)$, $(b:-a:0)$. The corresponding points on F_2^4 are $(0:0:0:c^2:-bc:b^2)$ and two others whose coordinates are similar combinations of the letters ca and ab. These points are in the plane of the conic on F_2^4 which corresponds to $la+mb+nc=0$. If we take a general point of *their* plane and apply equations (1) above we get $x_0':x_1':x_2':x_3':x_4':x_5'=a^2:ab:ac:b^2:bc:c^2$, which is the point on the Veronese surface that corresponds to the *point* $a:b:c$ in the plane of representation.

UNIVERSITY OF CALIFORNIA

ON COMPLETELY CONTINUOUS LINEAR TRANSFORMATIONS*

INGO MADDAUS

We shall deal with complete linear vector or so-called Banach spaces.† A completely continuous linear transformation is defined as a linear transformation which carries every bounded set into a compact set. In spaces of a finite number of dimensions, that is, spaces which are linear closed extensions of a finite number of elements, all bounded sets are compact.‡ Therefore singular transformations, that is to say, linear limited transformations which transform their domains into spaces of a finite number of dimensions, are completely continuous linear transformations. It is well known that the strong limit, or limit in the norm sense, of a sequence of completely continuous linear transformations is also completely continuous and linear.§ Consequently, the strong limit of a sequence of singular transformations is completely continuous and linear. The question naturally arises whether, conversely, every completely continuous linear transformation is the strong limit of a sequence of singular transformations.‖ This paper obtains a result for the domain of the transformation, a Banach space, and the range, a space to be defined and hereafter to be referred to as of type A. It will be seen that the conception of a space of type A is really a generalization of the idea of a Banach space with a denumerable base,¶ which will hereafter be referred to as of type S.

By a space of type A we shall mean a Banach space in which there exists a linearly independent sequence $\{f_n\}$ of elements of unit norm and a double sequence $\{L_{mn}(g)\}$ of linear limited operators such that for every g

$$\lim_{m=\infty} \left\| g - \sum_{n=1}^{m_n} L_{mn}(g) f_n \right\| = 0. \tag{1}$$

* Presented to the Society, September 10, 1937.

† Banach, *Théorie des Opérations Linéaires*, p. 53.

‡ Riesz, Acta Mathematica, vol. 41 (1927), p. 77.

§ Banach, loc. cit., p. 96.

‖ Hildebrandt, this Bulletin, vol. 37 (1931), p. 196.

¶ By a space of type S we shall mean a Banach space with a finite or denumerably infinite set of elements $\{f_i\}$ of unit norm such that every element g may be uniquely represented in the form $g=\sum_{i=1}^{\infty} c_i(g)f_i$, or $\lim_{n=\infty}\left\|g-\sum_{i=1}^{n} c_i(g)f_i\right\|=0$, where for a fixed index i the coefficients $c_i(g)$ are bounded linear operators on the space. See Schauder, Mathematische Zeitschrift, vol. 26 (1927), p. 47, and Banach, loc. cit., p. 110.

It is not required that there be only one double sequence $\{L_{mn}\}$ satisfying relation (1) for a given space.

Obviously a space of type S is also of type A. It is interesting to note that the set of spaces of type A also includes as a subset spaces which have an integral representation instead of the series representation of the spaces of type S. We shall designate such spaces by E and define them as follows. A space of type E is a Banach space in which there is a set of elements $f(t)$, $(0 \leqq t \leqq 1)$, of the power of the continuum, each of unit norm, such that every element g of E is of the form

$$g = \int_0^1 f(t)d_t\lambda(g, t),$$

or

$$\lim_{N_\sigma \to 0} \left\| g - \sum_\sigma f(\tau_i)\{\lambda(g, t_i) - \lambda(g, t_{i-1})\} \right\| = \lim_{N_\sigma \to 0} \left\| g - \sum_\sigma f(\tau_i)\Delta_i\lambda(g) \right\| = 0,$$

where the points $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ are elements of a partition σ of the interval (0, 1) and $t_{i-1} \leqq \tau_i \leqq t_i$. For each value of t, $\lambda(g, t)$ will be a bounded linear operator on the space, whence the same may be said of $\Delta_i\lambda(g)$ for each value of i. It will be understood that the integral is taken in the Riemann sense, and we shall mean by the norm of σ, written N_σ, the maximum of the lengths of the intervals formed by the partition σ.

To show that a space E is of type A it is only necessary to choose a sequence of subdivisions $\{\sigma_n\}$ such that $N_{\sigma_n} \leqq 1/n$ and a sequence of points τ_{in}, $(i = 1, 2, \cdots, n)$, such that

$$\lim_{N_{\sigma_n} \to 0} \left\| g - \sum_{\sigma_n} f(\tau_{in})\Delta_{in}\lambda(g) \right\| = 0,$$

for every g, where the second subscript after both τ and Δ refers to the partition σ_n.

It may easily be shown that every space of type S is of type E as well as of type A.

Because the spaces of type A include those of types S and E as special cases, we shall confine our attention to the former.

For each value of m, $T_m(g) = g - \sum_{n=1}^{m_n} L_{mn}(g)f_n$ is a linear limited transformation on a space A of type A. This follows directly from the linear limitedness of each $L_{mn}(g)$. We then have this lemma:

LEMMA. *The sequence* $\{T_m(g)\}$ *of linear limited transformations is such that* $\lim_{m=\infty}\|T_m(g)\|=0$ *uniformly on every self-compact partial set* H *of* A.

From the linear limitedness of each $T_m(g)$ and the fact that $\lim_{m=\infty}\|T_m(g)\|=0$ for each g of A, it follows that there exists an $M>0$, independent of both g and m, such that $\|T_m(g)\|<M\|g\|$.* By the total boundedness of the self-compact set H of E,† if $e>0$ is arbitrarily chosen then there is a finite set $g_1, g_2, \cdots, g_p$ of H such that every g of H is interior to at least one of the spheres of centers g_i, $(i=1, 2, \cdots, p)$, and of radius e/M. Obviously $\|T_m(g)\|$ approaches zero uniformly on the finite set g_i, $(i=1, 2, \cdots, p)$, so that for $n\geqq m(e)$ we have $\|T_n(g)\|\leqq e$ on this set. Then for any g of H and some g_i,

$$|\,\|T_m(g)\|-\|T_m(g_i)\|\,|\leqq\|T_m(g-g_i)\|\leqq M\|g-g_i\|\leqq e,$$

whence $\|T_n(g)\|\leqq e$ for each g of H when $n\geqq m(e)$. This proves our lemma.

THEOREM. *Every completely continuous linear transformation of a Banach space into a space of type A is the strong limit of a sequence of singular transformations.*

Let U be a completely continuous linear transformation of a Banach space D to a space A of type A with the base $\{f_n\}$, and let $g=U(x)$, where x is of D. Then $U_m(x)=\sum_{n=1}^{m_n}L_{mn}(g)f_n$ is a singular transformation of D into the linear closed extension of the finite number of elements f_n, $(n=1, 2, \cdots, m_n)$, which forms a subset of A.

Let A' be the transform by U of those elements D' of D whose norms are less than or equal to unity. Since U is completely continuous the set A' is self-compact, and by the previous lemma, $\|g-\sum_{n=1}^{m_n}L_{mn}(g)f_n\|$ approaches zero uniformly on A', consequently its equal $\|U(x)-U_m(x)\|$ must do likewise on D'. Hence the norm of the difference between the completely continuous linear transformation U and the singular transformation U_m approaches zero with $1/m$. This completes the proof.

From the above theorem and the observations of the introduction it follows directly that a linear transformation of a Banach space to a space of type A is completely continuous if and only if it is the strong limit of a sequence of singular transformations.

Since Hilbert space, the space of all continuous functions on a finite

* Banach, loc. cit., p. 80, Theorem 5.

† Hahn, *Reelle Funktionen*, 1921, p. 89.

closed interval with norm the absolute value of the function, and the space of all functions which are Lebesgue integrable to the pth power, $p \geqq 1$, with norm the pth root of the integral of the pth power of the absolute value of the function, are all spaces with a denumerable base in the sense of Schauder and Banach, and consequently of type A, the above theorem holds of all completely continuous linear transformations with Banach spaces as domains and such spaces as ranges.*

UNIVERSITY OF MICHIGAN

MULTIVALENT FUNCTIONS OF ORDER p†

M. S. ROBERTSON‡

1. **Introduction.** For the class of k-wise symmetric functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1, \ a_n = 0 \text{ for } n \not\equiv 1 \pmod{k}, \tag{1.1}$$

which are regular and univalent within the unit circle, it has been conjectured that there exists a constant $A(k)$ so that for all n

$$|a_n| \leqq A(k) n^{2/k-1}. \tag{1.2}$$

Proofs of this inequality for $k=1, 2, 2, 3$, were given by J. E. Littlewood,§ R. E. A. C. Paley and J. E. Littlewood,‖ E. Landau,¶ and V. Levin** respectively. As far as the author is aware there is no valid proof†† for $k>3$ in the literature as yet.

It is the purpose of this note to point out that the methods of proof

* Hildebrandt, this Bulletin, vol. 36 (1931), p. 197.

† Presented to the Society, February 20, 1937.

‡ The author is indebted to the referee for helpful suggestions which led to a revision of this note.

§ See J. E. Littlewood, *On inequalities in the theory of functions*, Proceedings of the London Mathematical Society, (2), vol. 23 (1925), pp. 481–519.

‖ See R. E. A. C. Paley and J. E. Littlewood, *A proof that an odd schlicht function has bounded coefficients*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 167–169.

¶ See E. Landau, *Über ungerade schlichte Funktionen*, Mathematische Zeitschrift, vol. 37 (1933), pp. 33–35.

** See V. Levin, *Ein Beitrag zum Koeffizientproblem der schlichten Funktionen*, Mathematische Zeitscrift, vol. 38 (1934), pp. 306–311.

†† See K. Joh and S. Takahashi, *Ein Beweis für Szegösche Vermutung über schlichte Potenzreihen*, Proceedings of the Imperial Academy of Japan, vol. 10 (1934) pp. 137–139. The proof therein was found to be defective: see Zentralblatt für Mathematik, vol. 9 (1934), pp. 75–76.

used in obtaining the inequality (1.2) for $k=2$ can be utilized to obtain a more general inequality for functions multivalent of order p with respect to the unit circle, provided these functions in question have no zeros within the unit circle other than at the origin. More specifically, let m be a non-negative integer, and let

$$(1.3) \qquad f(z) = \sum_{1+mk}^{\infty} a_n z^n, \qquad a_{1+mk} = 1,\ a_n = 0 \text{ for } n \not\equiv 1 \pmod{k},$$

be a k-wise symmetric function, regular and p-valent within the unit circle with $f(z) \neq 0$ for $0 < |z| < 1$. Then for all n

$$(1.4) \qquad |a_n| < A(p, k) n^{2p/k-1}, \qquad k < 4p,$$

where $A(p, k)$ is a constant independent of n and $f(z)$. We note in passing that a sufficient condition that $f(z) \neq 0$ for $0 < |z| < 1$ is that $p < k(m+1)+1$. For, if $f(z)$ vanishes in $0 < |z| < 1$ it vanishes $(1+mk)+k$ times within the unit circle for the reason that $f(z)$ is k-wise symmetric. On the other hand, $f(z)$ cannot vanish more than p times within the unit circle as $f(z)$ is p-valent. The condition $f(z) \neq 0$ for $0 < |z| < 1$ is automatically fulfilled for the following special case which is a generalization of the Paley-Littlewood inequality ($p=1$). Let the $2p$-wise symmetric function

$$(1.5) \qquad f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1,\ a_n = 0 \text{ for } n \not\equiv 1 \pmod{2p}$$

be regular and multivalent of order p within the unit circle; then the coefficients are uniformly bounded.

The inequality (1.4) for the particular case $k=1$, but general p, was shown to be true by M. Biernacki, who found that if the condition $f(z) \neq 0$ for $0 < |z| < 1$ be discarded, the inequality for the coefficients takes the form

$$(1.6) \qquad |a_n| < A(p) \cdot \mu_q n^{2p-1},$$

where $\mu_q = \text{maximum } \{|a_1|, |a_2|, \cdots, |a_q|\}$, q being the number of zeros of $f(z)$ located within the unit circle.*

2. **The proof for** (1.4). The following lemma for $p=1$ has been used by many authors, and is no doubt well known to many for $p>1$, though I know of no place in the literature where it has been proved.

* See M. Biernacki, *Sur les fonctions multivalentes d'ordre p*, Paris Comptes Rendus, vol. 204 (1936), pp. 449–451.

LEMMA. *Every function $f(z)$ which is k-wise symmetric, regular, and multivalent of order p in the unit circle can be represented in the form*

$$f(z) = \{F(z^k)\}^{1/k}, \tag{2.1}$$

where $F(z)$ is regular and multivalent of order p in the unit circle. Conversely, if $F(z)$ is any function regular and multivalent of order p in the unit circle, and if $F(z) \neq 0$ for $0 < |z| < 1$, then $f(z)$ is also multivalent of order p and regular in the unit circle.

PROOF. We may assume $f(z)$ is given by (1.3) and define $F(z)$ to be

$$F(z) = z\left(\sum_{1+mk}^{\infty} a_n z^{(n-1)/k}\right)^k$$

which is regular within the unit circle. Thus (2.1) holds.

Since $f(z)$ is p-valent, then there is a value $te^{i\phi}$, $(t>0)$, so that $f(z_i) = te^{i\phi}$ for exactly p distinct values z_i, $(i=1, 2, \cdots, p)$. Moreover, since $f(z)$ is k-wise symmetric one has

$$f(z_i e^{2s\pi i/k}) = e^{2s\pi i/k} \cdot f(z_i) = te^{(2s\pi/k+\phi)i}$$

for $s=0, 1, 2, \cdots, (k-1)$. Thus $f^k(z_i') = t^k e^{k\phi i}$ for $z_i' = z_i e^{2s\pi i/k}$, that is for pk values z_i' of z. Thus $f^k(z)$, and consequently $F(z^k)$, are pk-valent. Hence $F(z)$ is p-valent. The converse may be proved similarly.

In the proof below we shall make use of an inequality of M. Cartwright* for multivalent functions $F(z)$ of order p in the unit circle. We assume here that $F(z) \neq 0$ for $0 < |z| < 1$ so that we have

$$|F(re^{i\theta})| < A(p)(1-r)^{-2p}. \tag{2.2}$$

The method of proof for (1.4) now is that of E. Landau† (for $p=1$, $k=2$) with but slight modifications to take care of $p>1$, $k \geqq 2$. Let $f(z)$ be defined as in (1.3), and $F(z)$ as in (2.1). Let $s=k+1$ and

$$\{F(z^s)\}^{1/s} = \sum_1^{\infty} c_n z^n = z^{1+mk} + \cdots,$$

$$\{F(z^{ks})\}^{1/ks} = \sum_1^{\infty} d_n z^n = z^{1+mk} + \cdots,$$

$$\sum_1^{\infty} a_n z^{ns} = \left(\sum_1^{\infty} d_n z^n\right)^s.$$

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* See M. Cartwright, *Some inequalities in the theory of functions*, Mathematische Annalen, vol. 111 (1935), pp. 98–118.

† See E. Landau, loc. cit. See also K. Joh and S. Takahashi, loc. cit.

By differentiation, we obtain

$$s\sum_1^\infty na_nz^{ns-1} = s\left(\sum_1^\infty c_nz^{nk}\right)\left(\sum_1^\infty nd_nz^{n-1}\right),$$

$$na_n = \sum_{k\mu+\nu=(k+1)n} c_\mu \nu d_\nu,$$

and we deduce the inequalities

$$n^2|\,a_n\,|^2 < (k+1)n \sum_{\mu\leqq 2n} |\,c_\mu\,|^2 \cdot \sum_{\nu\leqq(k+1)n} \nu\,|\,d_\nu\,|^2,$$

$$\sum_1^m n\,|\,c_n\,|^2r^{2n} \leqq p\left[\frac{A(p)}{(1-r^{k+1})^{2p}}\right]^{2/(k+1)} < A_1(p,k)(1-r)^{-4p/(k+1)},$$

$$\sum_1^m n\,|\,d_n\,|^2r^{2n} \leqq A_2(p,k)(1-r)^{-4p/ks}, \qquad s=k+1.$$

For $r=e^{-1/m}$, we then have for an arbitrary positive integer m

$$C_m = \sum_{n=1}^m n\,|\,c_n\,|^2 < A_3(p,k)m^{4p/(k+1)},$$

$$\sum_1^m n\,|\,d_n\,|^2 < A_4(p,k)m^{4p/ks}, \qquad s=k+1,$$

$$\sum_1^m |\,c_n\,|^2 = \sum_1^m \frac{C_n - C_{n-1}}{n} < A_5(p,k)m^{4p/(k+1)-1} \text{ for } k<4p-1,$$

$$n^2|\,a_n\,|^2 < (k+1)n\cdot A_5(p,k)(2n)^{4p/(k+1)-1}\cdot A_4(p,k)(\overline{k+1}\,n)^{4p/k(k+1)};$$

whence

$$|\,a_n\,| < A(p,k)n^{2p/k-1} \quad \text{for} \quad k<4p-1.$$

For $k=4p-1$ we may use the method of V. Levin* with the obvious modifications to take care of $p>1$. This will give (1.4) for $k=4p-1$. Thus (1.4) holds for $k<4p$.

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* See V. Levin, loc. cit.

ON THE ORDER OF THE PARTIAL SUMS OF A FOURIER SERIES*

W. C. RANDELS

We propose to show here that the known estimate $S_n = o(n)$ cannot be improved. To do this it is sufficient to show that there exists a sequence of functions $f_n(x)$ for which there is a constant A such that,

$$S_n(f_n, 0) > An, \tag{1}$$

and

$$S_\nu(f_n, 0) \to 0 \quad \text{as} \quad \nu \to \infty, \tag{2}$$

$$\int_{-\pi}^{\pi} | f_n(x)dx | < 1. \tag{3}$$

For, if (1), (2), and (3) are satisfied, then for every sequence of positive numbers d_n, with $d_n \to 0$ as $n \to \infty$, lim sup $nd_n = \infty$, we can choose a sequence of integers n_i such that

$$\left| \sum_{j=1}^{i-1} d_{n_j} S_{n_i}(f_{n_j}, 0) \right| < \frac{A}{3} n_i d_{n_i} \tag{4}$$

and

$$d_{n_{i+1}} < \frac{A}{3\pi} d_{n_i}. \tag{5}$$

We notice that

$$S_{n_j}(f_{n_i}, 0) = \frac{\pi}{2} \int_{-\pi}^{\pi} f_{n_i}(x) D_{n_j}(x) dx < \pi n_j \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx < \pi n_j, \tag{6}$$

and this implies that the constant A in (1) is less than π. Then, if $f(x)$ is defined by

$$f(x) = \sum_{i=1}^{\infty} d_{n_i} f_{n_i}(x),$$

$f(x) \in L$, since from (3) and (5)

$$\int_{-\pi}^{\pi} | f(x) | \, dx \leqq \sum_{i=1}^{\infty} d_{n_i} \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx < \frac{d_1 A}{\pi} \sum_{i=1}^{\infty} 3^{-i} = \frac{d_1 A}{2\pi}.$$

* Presented to the Society, April 10, 1937.

We have also

$$
\begin{aligned}
|S_{n_i}(f,0)| &= \left|\sum_{j=1}^{\infty} d_{n_j} S_{n_i}(f_{n_j},0)\right| \\
&= \left|\sum_{j=1}^{i-1} d_{n_j} S_{n_i}(f_{n_j},0) + d_{n_i} S_{n_j}(f_{n_i},0) + \sum_{j=i+1}^{\infty} d_{n_j} S_{n_i}(f_{n_i},0)\right| \\
&> A n_i d_{n_i} - \tfrac{1}{3} A n_i d_{n_i} - \pi n_i \sum_{j=i+1}^{\infty} d_{n_j} \\
&> \tfrac{1}{3} A n_i d_{n_i}.
\end{aligned}
$$

We shall now prove the existence of a sequence $f_n(x)$ with the properties (1), (2), and (3). We define

$$
f_n(x) = \begin{cases} n/2, & \pi/4(n+\tfrac{1}{2}) \leqq |x| \leqq \pi/2(n+\tfrac{1}{2}), \\ 0, & \text{elsewhere.} \end{cases}
$$

Then

$$
\int_{-\pi}^{\pi} |f_n(x)|\, dx = \frac{n\pi}{8(n+\tfrac{1}{2})} < 1, \tag{3}
$$

and, since $f_n(x)$ vanishes in the neighborhood of the origin,

$$
S_\nu(f_n, 0) \to 0 \quad \text{as} \quad \nu \to \infty. \tag{2}
$$

Finally,

$$
\begin{aligned}
S_n(f_n,0) &= \pi \int_0^{\pi} f_n(x) \frac{\sin(n+\tfrac{1}{2})x}{\sin x/2}\, dx \\
&= \frac{n\pi}{2} \int_{\pi(n+1/2)/4}^{\pi/2(n+1/2)} \frac{\sin(n+\tfrac{1}{2})x}{\sin x/2}\, dx \qquad (1) \\
&> \frac{\pi n}{2} \cdot \frac{\pi}{4(n+\tfrac{1}{2})} \cdot \frac{2(n+\tfrac{1}{2})}{\pi} > \pi n/4,
\end{aligned}
$$

and our theorem is proved.

We shall merely mention that similar results could be easily obtained for the Cesàro sums of a Fourier series. For the sequence of functions we have constructed satisfies (1), (2), and (3) when $S_n(f, x)$ is replaced by $S_n^{\alpha}(f, x)$ where $S_n^{\alpha}(f, x)$ is the C_α transform of the Fourier series of $f(x)$.

We shall now point out a consequence of this result.* We know, of

* The author is indebted to Professor Szász for pointing this out.

course, that the Fourier series of a function of bounded variation converges everywhere; we are now able to state that nothing can be said about the order of convergence even if we restrict ourselves to absolutely continuous functions. For let us suppose that there is some sequence c_n, $(c_n \to 0)$, such that for all absolutely continuous $F(x)$

$$S_n(F, x) - F(x) = O(C_n). \tag{7}$$

Let $f(x)$ be the function defined above, and define

$$f_a(x) = f(a + x) - f(a - x).$$

Then

$$f_a(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$$

and

$$S_{n_i}(f_a, a) > \frac{A}{3} n_i d_{n_i} - o(1) \neq o(n_i d_{n_i}). \tag{8}$$

If $F(x)$ is defined by

$$F(x) = \int_0^x f(x)dx,$$

we have

$$F(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx,$$

where $A_n = b_n/n$, $n \geqq 1$. Then, if (7) is true,

$$\begin{aligned} S_n(f_a, x) &= \sum_{\nu=1}^{n} \nu A_\nu \sin \nu x \\ &= [-S_0(F, x) + F(x)] \sin x + [S_n(F, x) - F(x)] n \sin nx \\ &\quad + \sum_{\nu=1}^{n-1} [S_\nu(F, x) - F(x)][\nu \sin \nu x - (\nu + 1) \sin (\nu + 1)x] \\ &= O(nC_n). \end{aligned}$$

But by (8) the d_n can be chosen so that this is impossible, and therefore (7) must be false.

NORTHWESTERN UNIVERSITY

ON SOME INEQUALITIES OF S. BERNSTEIN AND W. MARKOFF FOR DERIVATIVES OF POLYNOMIALS*

A. C. SCHAEFFER AND R. J. DUFFIN

A well known inequality on the derivatives of polynomials is that of S. Bernstein.†

BERNSTEIN'S THEOREM. *Suppose that $f(x)$ is a polynomial of degree n or less, and that in the interval $(-1, 1)$*

$$|f(x)| \leqq 1.$$

Then

$$|f'(x)|^2 \leqq \frac{n^2}{1-x^2}, \qquad -1 \leqq x \leqq 1,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$, $|\gamma| = 1$,‡ where $T_n(x)$ is the nth Tchebychef polynomial.

The extension of this theorem of Bernstein to the higher derivatives plays an important role in this paper. Thus, if $f(x)$ satisfies the conditions given in Bernstein's theorem, we obtain the inequality

$$|f^{(p)}(x)|^2 \leqq \left(\frac{d^p}{dx^p}\cos n\theta\right)^2 + \left(\frac{d^p}{dx^p}\sin n\theta\right)^2, \qquad x = \cos\theta,$$

for x in $(-1, 1)$. Using this inequality we are able to give a simple proof of W. Markoff's theorem,§ which states that under the conditions of Bernstein's theorem

$$|f^{(p)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2p-1)}, \qquad -1 \leqq x \leqq 1.$$

* Presented to the Society, April 8, 1938.

† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mémoires de l'Académie Royale de Belgique, (2), vol. 4 (1912), pp. 1–104. M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.

‡ γ stands hereafter for a constant, real or complex, of absolute magnitude 1.

§ W. Markoff, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–258, translated by J. Grossman. The original appeared in Russian in 1892.

G. Szegö, *Über einen Satz von A. Markoff*, Mathematische Zeitschrift, vol. 23 (1925) pp. 45–61.

Before proceeding with the proof of these inequalities we find it necessary to establish several lemmas concerning the properties of the Tchebychef polynomial $T_n(x)$, and a related function $S_n(x)$, (not a polynomial) which are defined by the relations

$$(1) \qquad \begin{aligned} T_n(x) &= \cos n\theta, \\ S_n(x) &= \sin n\theta, \end{aligned} \qquad x = \cos\theta,$$

n being any positive integer. These functions are independent solutions of the differential equation

$$(2) \qquad (1 - x^2)R''(x) - xR'(x) + n^2R(x) = 0,$$

the general real solution of which may be written

$$R(x) = aT_n(x) + bS_n(x) = c\cos(n\theta - \alpha), \qquad x = \cos\theta,$$

where a, b, c, and α are real constants. Differentiating (2) p times we obtain

$$(3) \quad (1 - x^2)R^{(p+2)}(x) - (2p + 1)xR^{(p+1)}(x) + (n^2 - p^2)R^{(p)}(x) = 0,$$

and this may be written in the equivalent form

$$(4) \quad \frac{d}{dx}\{[1 - x^2][R^{(p+1)}(x)]^2 + [n^2 - p^2][R^{(p)}(x)]^2\} = 4px[R^{(p+1)}(x)]^2.$$

For $p \leqq n$ the functions $T_n^{(p)}(x)$ and $S_n^{(p)}(x)$ are particular solutions of (3) and (4).

Let $M_p(x)$ be defined by the relation

$$(5) \qquad \begin{aligned} M_p(x) &= (T_n^{(p)}(x))^2 + (S_n^{(p)}(x))^2 \\ &\equiv \left(\frac{d^p}{dx^p}\cos n\theta\right)^2 + \left(\frac{d^p}{dx^p}\sin n\theta\right)^2. \end{aligned}$$

Then it is clear that $M_p(x)$ also satisfies (4), that is

$$(6) \qquad \frac{d}{dx}\{(1 - x^2)M_{p+1}(x) + (n^2 - p^2)M_p(x)\} = 4pxM_{p+1}(x).$$

LEMMA 1. *In the open interval* $(-1, 1)$, $T_n^{(p)}(x)$ *has* $n-p$ *zeros all of which are simple,* $(p=0, 1, 2, \cdots, n)$; $S_n^{(p)}(x)$ *has* $n+1-p$ *zeros all of which are simple,* $(p=1, 2, 3, \cdots, n+1)$; *and the zeros of* $T_n^{(p)}(x)$ *and* $S_n^{(p)}(x)$ *separate one another,* $(p=1, 2, 3, \cdots, n)$.

PROOF. From the definition (1) we see that $T_n(x)$ has n simple zeros in the interior of $(-1, 1)$, and since it is a polynomial of degree n, these are its only zeros. From Rolle's theorem it follows that

$T_n^{(p)}(x)$, $(p=1, 2, \cdots, n)$, has exactly $n-p$ zeros in $(-1, 1)$, all of which are simple. From (1) we see that $S_n(x)$ may be expressed in the form

$$S_n(x) = T_n'(x)(1 - x^2)^{1/2}/n,$$

from which it follows that $S_n(x)$ has $n+1$ zeros in the closed interval $(-1, 1)$, and that $S_n'(x)$ becomes infinite as $x \to \pm 1$. By successive use of Rolle's theorem it follows that $S_n^{(p)}(x)$ must have at least $n+1-p$ distinct zeros in the open interval $(-1, 1)$. But $T_n^{(p)}(x)$ and $S_n^{(p)}(x)$ are linearly independent solutions of the differential equation (3), so by Sturm's well known theorem the zeros of $T_n^{(p)}(x)$ and $S_n^{(p)}(x)$ separate one another. It follows that $S_n^{(p)}(x)$ has exactly $n+1-p$ zeros in the interval $(-1, 1)$, all of which are simple, $(p=1, 2, 3, \cdots, n)$. For the case $p=n+1$ we see from (3) that $S_n^{(n+1)}(x)$ is a solution of the differential equation

$$(1 - x^2)\frac{d}{dx}S_n^{(n+1)}(x) - (2n + 1)xS_n^{(n+1)}(x) = 0.$$

Suppose that $S_n^{(n+1)}(x)$ had at an interior point x_0 a zero of order k, $(k \geqq 1)$. Then, from the differential equation, $dS_n^{(n+1)}(x)/dx$ would have a zero of order k (or of order $k+1$ if $x_0=0$). This would demand that $S_n^{(n+1)}(x)$ have a zero of order $k+1$ (or of order $k+2$ if $x_0=0$), in contradiction to the assumption. This proves that $S_n^{(n+1)}(x)$ cannot have a zero.

LEMMA 2. *In the expansion*

$$M_p(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k},$$

valid for $|x| < 1$, $(1 \leqq p \leqq n)$, *all the coefficients are greater than zero.*

PROOF. The statement is evidently true for $p=1$, since, by definition,

$$M_1(x) = (T_n'(x))^2 + (S_n'(x))^2 = n^2/(1 - x^2).$$

We now proceed by induction. Let

$$M_p(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k},$$

$$M_{p+1}(x) = \sum_{k=0}^{\infty} b_{2k}x^{2k}.$$

On substituting these power series in the differential equation (6) we obtain between the coefficients the relation

$$kb_{2k} + k(n^2 - p^2)a_{2k} = (2p + k)b_{2k-2}. \tag{7}$$

Suppose that every coefficient a_{2k} is greater than zero. From the relation (7) we see that if, for a particular k, b_{2k} is positive, then b_{2k-2} is also positive; and by repeated use of this relation each of the coefficients b_{2k-4}, b_{2k-6}, $\cdots$, b_2, b_0 is positive. Thus if b_{2k} is greater than zero, then the same is true of each of the preceding coefficients. The functions $S_n^{(p+1)}(x)$ and $M_{p+1}(x)$ are unbounded in the interval $(-1, 1)$ since $S_n'(x)$ becomes infinite as $x \to \pm 1$. This shows, since $M_{p+1}(x)$ is non-negative, that there are arbitrarily large values of k for which b_{2k} is greater than zero. But if one coefficient is positive, we have shown that all the preceding ones are also positive, and it follows that all b_{2k} are positive. This completes the induction.

By definition (1), $T_n(1) = 1$, and by (3)

$$(2p + 1)T_n^{(p+1)}(1) = (n^2 - p^2)T_n^{(p)}(1).$$

From this we find by induction that

$$T_n^{(p)}(1) = \frac{n^2(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - (p-1)^2)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2p - 1)}. \tag{8}$$

THEOREM 1. *Let $f(x)$ be a polynomial of degree n or less with real coefficients such that*

$$|f(x)| \leqq 1, \qquad -1 \leqq x \leqq 1,$$

and suppose that $f(x) \not\equiv \gamma T_n(x)$. Then for every real α the first n derivatives of the function

$$\cos \alpha T_n(x) + \sin \alpha S_n(x) - f(x)$$

can have only simple zeros in the interval $(-1, 1)$.

PROOF. There is no loss of generality in supposing that $0 \leqq \alpha < \pi$. Let

$$\begin{cases} R(x) = \cos \alpha T_n(x) + \sin \alpha S_n(x) = \cos(n\theta - \alpha), & x = \cos\theta \\ R'(x) = n \sin(n\theta - \alpha)/\sin\theta. \end{cases} \tag{9}$$

Then at the points where $R(x)$ vanishes we have

$$(R'(x))^2 = \frac{n^2}{1 - x^2};$$

hence, by Bernstein's theorem,

$$|f'(x)| < |R'(x)| \tag{10}$$

at these points. If $\alpha \neq 0$, this inequality is also true in a neighborhood of the points -1 and $+1$, for $S_n'(x)$ becomes infinite here while $T_n'(x)$ and $f'(x)$ are bounded in the whole interval. We now distinguish two cases, $\alpha = 0$ and $\alpha > 0$.

CASE I. $\alpha = 0$.

In this case $R(x) \equiv T_n(x)$, and the function considered reduces to

$$T_n(x) - f(x)$$

which is a polynomial of degree n or less, not identically zero. Then by (10)

$$T_n'(x) - f'(x)$$

is alternately plus and minus at the n points where $R(x)$ vanishes; so it has at least $n-1$ distinct zeros. Using Rolle's theorem we see that $T_n^{(p)}(x) - f^{(p)}(x)$ has at least $n-p$ distinct zeros, $(1 \leqq p \leqq n)$, and it can have no others, as it is a polynomial of degree $n-p$. Thus all its zeros are simple and Theorem 1 is true for the case $\alpha = 0$.

CASE II. $0 < \alpha < \pi$.

We are going to show first that in this case the function

$$R'(x) - f'(x)$$

has at least n distinct zeros.

If $\alpha = \pi/2$, then $R(x) \equiv S_n(x)$, and we see that at the $n-1$ zeros of $S_n(x)$ in the interior of $(-1, 1)$ and in a neighborhood of the two end points the inequality (10) is satisfied. Thus, $S_n'(x)$ is alternately plus and minus at $n+1$ successive points where (10) is true; so the function

$$R'(x) - f'(x) \equiv S_n'(x) - f'(x)$$

has at least n distinct zeros in $(-1, 1)$.

If $\alpha \neq \pi/2$, then either $0 < \alpha < \pi/2$ or $\pi/2 < \alpha < \pi$, and as the two cases are essentially the same we shall consider only the case $0 < \alpha < \pi/2$. From relation (9) we see that $R(x)$ vanishes at the n points $\theta = [(k+\frac{1}{2})\pi + \alpha]/n$, $(k = 0, 1, \cdots, n-1)$, in $(-1, 1)$. Selecting these n points and one point from a small neighborhood of $x = 1$, we have $n+1$ points in $(-1, 1)$ at which the inequality (10) is satisfied. From the relation (9) it is seen that $R'(x)$ is alternately plus and minus at these $n+1$ successive points. Then the function

$$R'(x) - f'(x)$$

is alternately plus and minus at $n+1$ points and so has at least n distinct zeros.

Thus if $0<\alpha<\pi$, the function

$$R'(x) - f'(x)$$

has at least n distinct zeros in $(-1, 1)$. Using Rolle's theorem one shows that

$$R^{(p)}(x) - f^{(p)}(x)$$

has at least $n+1-p$ distinct zeros. If it had one multiple zero, then $R^{(p+1)}(x)-f^{(p+1)}(x)$ would have at least $n+1-p$ distinct zeros, and finally

$$R^{(n+1)} - f^{(n+1)}$$

would have at least one zero. Since $f(x)$ and $T_n(x)$ are polynomials of degree n at most this means that $S_n^{(n+1)}(x)$ has at least one zero at some interior point of $(-1, 1)$; but by Lemma 1 this is impossible. This completes the proof of Theorem 1.

THEOREM 2. *For any polynomial $f(x)$ of degree n or less, the inequality $|f(x)| \leqq 1$ in $(-1, 1)$ implies*

$$|f^{(p)}(x)|^2 \leqq M_p(x), \qquad -1 \leqq x \leqq 1;\ p = 1, 2, 3, \cdots, n,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$.

PROOF. For simplicity we shall suppose that $f(x) \not\equiv \gamma T_n(x)$. We shall consider first the case in which all the coefficients of $f(x)$ are real. Suppose that at some point x_0, $(-1<x_0<1)$, we have

$$|f^{(p)}(x_0)|^2 \geqq M_p(x_0) \tag{11}$$

for some p, $(1 \leqq p \leqq n)$. Consider the function

$$R(x) + \lambda f(x),$$

where $R(x)$ is of the form

$$R(x) = \cos \alpha T_n(x) + \sin \alpha S_n(x), \tag{12}$$

and α and λ are real constants to be determined. We shall show that if (11) is true, then real α and λ exist, $(-1 \leqq \lambda \leqq 1)$, so that at the point x_0 the pth derivative of the function $R(x)+\lambda f(x)$ has a double zero.

Let α be chosen so that the relation

$$R^{(p+1)}(x_0)f^{(p)}(x_0) - R^{(p)}(x_0)f^{(p+1)}(x_0) = 0 \tag{13}$$

is satisfied. This is always possible for, on expressing $R^{(p)}(x)$ and $R^{(p+1)}(x)$ by use of equation (12), the relation (13) is equivalent to an equation of the form

$$a \cos \alpha + b \sin \alpha = 0,$$

where a and b are real, and this has a solution.

Having chosen α we see from Cauchy's inequality that

$$| R^{(p)}(x) |^2 \leqq \{\cos^2 \alpha + \sin^2 \alpha\} \{(T_n^{(p)}(x))^2 + (S_n^{(p)}(x))^2\} \equiv M_p(x).$$

Hence (supposing that inequality (11) is true) we choose λ, $(-1 \leqq \lambda \leqq 1)$, so that at the point x_0

$$R^{(p)}(x_0) + \lambda f^{(p)}(x_0) = 0.$$

Substituting this in equation (13), we have

$$f^{(p)}(x_0) \{ R^{(p+1)}(x_0) + \lambda f^{(p+1)}(x_0) \} = 0$$

and this means that the second factor is zero. Thus the pth derivative of the function

$$R(x) + \lambda f(x)$$

has a double zero at the point x_0, but by Theorem 1 this is impossible. The contradiction proves Theorem 2 in the case in which all the coefficients of $f(x)$ are real.

Now allow $f(x)$ to have complex coefficients,* and choose a real constant β so that, at a point x_0 arbitrarily chosen in $(-1, 1)$,

$$e^{i\beta} f^{(p)}(x_0)$$

is real. Writing

$$e^{i\beta} f(x) = f_1(x) + i f_2(x),$$

where $f_1(x)$ and $f_2(x)$ have real coefficients, we see that if $f_1(x)$ were of the form $f_1(x) \equiv \gamma T_n(x)$, then $f_2(x)$ would vanish at the $n+1$ points where $T_n(x) = \pm 1$ and so would vanish identically. Then $f(x)$ itself would be of the form $f(x) \equiv \gamma T_n(x)$; but we have supposed that this is not the case, so $f_1(x) \not\equiv \gamma T_n(x)$. We have already proved that Theorem 1 applies to the polynomial $f_1(x)$, and since $e^{i\beta} f^{(p)}(x_0) = f_1^{(p)}(x_0)$, it follows that

$$| f^{(p)}(x_0) |^2 = | f_1^{(p)}(x_0) |^2 < M_p(x_0).$$

* This method of extending inequalities to polynomials with complex coefficients has been used by S. Bernstein, *Leçons sur les propriétés extrémals et meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, 1926, p. 45.

This completes the proof of Theorem 2.

LEMMA 3. *Let $f(x)$ be a polynomial of degree n or less, such that*

$$|f(x)| \leqq 1$$

in $(-1, 1)$. If x_0 is a point of $(-1, 1)$ which lies either to the left or to the right of all zeros of $S_n^{(p)}(x)$, then

$$|f^{(p)}(x_0)| \leqq |T_n^{(p)}(x_0)|,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$.

PROOF. Let a_k, $(k=1, 2, 3, \cdots, n+1-p)$, be the zeros of $S_n^{(p)}(x)$ in the interval $(-1, 1)$. Let $\phi(x)$ be a polynomial of degree $n+1-p$ which vanishes at these $n+1-p$ points,

$$\phi(x) = (x - a_1)(x - a_2) \cdots (x - a_{n+1-p}).$$

Then using the Lagrange interpolation formula we have, since $f^{(p)}(x)$ is a polynomial of degree $n-p$ or less,

$$f^{(p)}(x) = \phi(x) \sum_{k=1}^{n+1-p} \frac{f^{(p)}(a_k)}{\phi'(a_k)(x - a_k)},$$

and there is a similar expression for $T_n^{(p)}(x)$. The zeros of $S_n^{(p)}(x)$ and $T_n^{(p)}(x)$ interspace one another, so $T_n^{(p)}(x)$ is alternately plus and minus at the successive zeros of $S_n^{(p)}(x)$, and it is easily seen that $\phi'(x)$ alternates in sign at successive zeros of $\phi(x)$. Thus all the numbers

$$\frac{T_n^{(p)}(a_k)}{\phi'(a_k)}$$

are of the same sign, $(k=1, 2, 3, \cdots, n+1-p)$. Moreover, by Theorem 2, we have at the zeros of $S_n^{(p)}(x)$

$$|f^{(p)}(a_k)|^2 \leqq M_p(a_k) = (T_n^{(p)}(a_k))^2. \tag{14}$$

Now let x_0 be a point which lies to the right of all zeros of $S_n^{(p)}(x)$, so that $x_0 - a_k > 0$, $(k = 1, 2, \cdots, n+1-p)$. Then it follows immediately that

$$\begin{aligned} |f^{(p)}(x_0)| &\leqq |\phi(x_0)| \sum \left| \frac{f^{(p)}(a_k)}{\phi'(a_k)(x_0 - a_k)} \right| \\ &\leqq |\phi(x_0)| \sum \left| \frac{T_n^{(p)}(a_k)}{\phi'(a_k)(x_0 - a_k)} \right| = |T_n^{(p)}(x_0)|. \end{aligned} \tag{15}$$

The equality can occur throughout (15) only if the equality is true

in (14), and by Theorem 2 this is true only if $f(x) \equiv \gamma T_n(x)$. This proves Lemma 3 in the case where x_0 lies to the right of all zeros of $S_n^{(p)}(x)$, and the same method is available if x_0 lies to the left of all zeros of $S_n^{(p)}(x)$.

We can now prove the theorem of W. Markoff.

MARKOFF'S THEOREM. *If $f(x)$ is a polynomial of degree n or less, the inequality $|f(x)| \leqq 1$, $(-1 \leqq x \leqq 1)$, will imply*

$$|f^{(p)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2p-1)}, \quad -1 \leqq x \leqq 1,$$

for $p = 1, 2, 3, \cdots, n$. The equality can occur only at $x = \pm 1$ and here only if $f(x) \equiv \gamma T_n(x)$.

PROOF. By Theorem 2 we see that

$$|f^{(p)}(x)|^2 \leqq (T_n^{(p)}(x))^2 + (S_n^{(p)}(x))^2 \equiv M_p(x),$$

and in Lemma 3 we have shown that $M_p(x)$ increases monotonically in (0, 1). Let a, $(|a| < 1)$, be the zero farthest to the right of $S_n^{(p)}(x)$ in the interval $(-1, 1)$; then in the interval $(-a, a)$ we have

$$|f^{(p)}(x)|^2 \leqq M_p(x) \leqq M_p(a) = (T_n^{(p)}(a))^2. \tag{16}$$

We know that $T_n^{(p)}(x)$ increases monotonically in the interval $(a, 1)$, since, by Lemma 1, its zeros lie in the interval $(-a, a)$. Moreover $T_n^{(p)}(x)$ is either an odd or an even function of x, hence the maximum of its absolute magnitude in the intervals $(-1, -a)$ and $(a, 1)$ occurs at $x = \pm 1$. Then if x_1 lies in $(-a, a)$,

$$|f^{(p)}(x_1)| \leqq |T_n^{(p)}(a)| < |T_n^{(p)}(1)|.$$

If x_2 lies in either of the intervals $(-1, -a)$, $(a, 1)$, we have from Lemma 3

$$|f^{(p)}(x_2)| \leqq |T_n^{(p)}(x_2)| \leqq |T_n^{(p)}(1)|,$$

and the equality can occur only at $x_2 = \pm 1$. Thus, if x lies in the interval $(-1, 1)$, we have the inequality

$$|f^{(p)}(x)| \leqq |T_n^{(p)}(1)|,$$

and the equality can occur only at $x = \pm 1$, and here only if $f(x) \equiv \gamma T_n(x)$. The explicit expression for $T_n^{(p)}(1)$ obtained in (8) furnishes the final step in the proof of Markoff's theorem.

PURDUE UNIVERSITY

SOME THEOREMS ON SUBSEQUENCES†

HUGH J. HAMILTON

It is obvious that, for any real sequence for which the sum Σ of the moduli of its elements exists and is finite, there exists a subsequence such that the modulus of the sum of its elements is not less than $\Sigma/2$. The purpose of this paper is to formulate and investigate analogous statements for complex sequences.

Let $\mathfrak{A}$ be the class of sequences, finite or infinite, $\{a_k\}$ (denoted alternatively by A) of non-zero complex numbers for which $\sum|a_k|<\infty$, and $\{a_j'\}$ (denoted alternatively by S), the general subsequence of $\{a_k\}$ for fixed $\{a_k\}$. Let $\mathfrak{B}$ be the class of sequences $\{b_k\}$ (denoted alternatively by B) of non-zero complex numbers for which $\sum|b_k|=\infty$, and $\{b_j'\}$ (denoted alternatively by T), the general subsequence of $\{b_k\}$ for fixed $\{b_k\}$.

The following facts will be established: (i) Given any sequence $\{a_k\}\epsilon\mathfrak{A}$, there then exists a subsequence $\{a_j^*\}$ for which $|\sum a_j^*| = \sup_S|\sum a_j'|$. (ii) If $\rho\equiv\inf_A \max_S|\sum a_j'|/\sum|a_k|$, then $\rho=1/\pi$. (iii) No sequence $\{a_k\}\epsilon\mathfrak{A}$ exists for which $\max_S|\sum a_j'|/\sum|a_k|=\rho$. (iv) Given any sequence $\{b_k\}\epsilon\mathfrak{B}$, there exists a subsequence $\{b_j^*\}$ such that‡

$$\limsup|\sum{}'b_j^*|/\sum_1^N|b_k| = \sup_T\limsup_N|\sum{}'b_j'|/\sum_1^N|b_k| = \limsup_N\sup_T|\sum{}'b_j'|/\sum_1^N|b_k| = \limsup_N\max_T|\sum{}'b_j'|/\sum_1^N|b_k|.$$

(v) If $\sigma=\inf_B\max_T\limsup_N|\sum'b_j'|/\sum_1^N|b_k|$, then $\sigma=\rho$. (vi) There exists a sequence $\{b_k\}\epsilon\mathfrak{B}$ for which $\max_T\limsup_N|\sum'b_j'|/\sum_1^N|b_k|=\sigma$.

Use will be made of abbreviations of the following sort: $A_k\equiv|a_k|$, $\phi_k\equiv\arg a_k$. For definiteness, the function "arg" will mean, throughout this paper, principal argument. Given any sequence $\{a_k\}\epsilon\mathfrak{A}$, define

$$F(\phi)\equiv\sum_{\cos(\phi-\phi_k)>0}A_k\cos(\phi-\phi_k) = \sum A_k\{\cos(\phi-\phi_k)+|\cos(\phi-\phi_k)|\}/2, \qquad 0\leqq\phi\leqq 2\pi.$$

† Presented to the Society, November 27, 1937.

‡ The notation $\sum'$ indicates summation over precisely those elements of the subsequence which occur among the elements of the original sequence summed elsewhere in the formula.

Being continuous, $F(\phi)$ attains its supremum. In what follows, to and including Theorem 3, $\{a_k\}$ will signify an arbitrary but fixed sequence of class $\mathfrak{A}$.

THEOREM 1. *Let ϕ^* be such that $F(\phi^*)=\max F(\phi)$, and let $\{a_j^*\}$ be the sequence of those elements of $\{a_k\}$ for which $\cos(\phi^*-\phi_k)>0$. Then $\sup_S|\sum a_j'|=F(\phi^*)=|\sum a_j^*|$.*

PROOF. Let $\{a_j'\}$ be any subsequence of $\{a_k\}$, and define $\phi=\arg\sum a_j'$. Then

$$|\sum a_j^*| \geqq \sum A_j^* \cos(\phi^*-\phi_j^*) = F(\phi^*) \geqq F(\phi)$$
$$= \sum_{\cos(\phi-\phi_k)>0} A_k \cos(\phi-\phi_k) \geqq \sum A_j' \cos(\phi-\phi_j') = |\sum a_j'|.$$

This establishes (i).

COROLLARY 1.1. *In the notation of Theorem 1, $\phi^*=\arg\sum a_j^*$.*

PROOF. Taking $\{a_j'\}\equiv\{a_j^*\}$ in the inequalities of Theorem 1, we see that $|\sum a_j^*|=\sum A_j^*\cos(\phi^*-\phi_j^*)$. That is, the modulus of $\sum a_j^*$ is equal to that of its projection on the ray of angle ϕ^*.

The following theorem and its corollary provide a sort of converse or dual of Theorem 1 and Corollary 1.1:

THEOREM 2. *Let $\{\bar a_j\}$ be a subsequence of $\{a_k\}$ for which $|\sum\bar a_j|=\max_S|\sum a_j'|$, and let $\bar\phi\equiv\arg\sum\bar a_j$. Then $\max F(\phi)=|\sum\bar a_j|=F(\bar\phi)$.*

PROOF. Let ϕ be any angle, $(0\leqq\phi\leqq2\pi)$, and $\{a_j'\}$ the sequence of those elements of $\{a_k\}$ for which $\cos(\phi-\phi_k)>0$. Then

$$F(\bar\phi) = \sum_{\cos(\bar\phi-\phi_k)>0} A_k\cos(\bar\phi-\phi_k) \geqq \sum \bar A_j\cos(\bar\phi-\bar\phi_j)$$
$$= |\sum\bar a_j| \geqq |\sum a_j'| \geqq \sum_{\cos(\phi-\phi_k)>0} A_k\cos(\phi-\phi_k) = F(\phi).$$

COROLLARY 2.1. *In the notation of Theorem 2, $\{\bar a_j\}$ is the sequence of those elements of $\{a_k\}$ for which $\cos(\bar\phi-\phi_k)>0$.*

PROOF. Taking $\phi=\bar\phi$ in the inequalities of Theorem 2, we see that

$$\sum_{\cos(\bar\phi-\phi_k)>0} A_k\cos(\bar\phi-\phi_k) = \sum\bar A_j\cos(\bar\phi-\bar\phi_j).$$

In conjunction with Theorem 3 (below), this proves the assertion.

THEOREM 3. *In the notation of Theorem 2, there exists no element a_κ of $\{a_\kappa\}$ for which $\cos(\bar\phi-\phi_\kappa)=0$.*

PROOF. If there were such an element, then $|\sum \bar{a}_j \pm a_\kappa| > |\sum \bar{a}_j|$, so that addition of a_κ to $\{\bar{a}_j\}$, if it were not already therein contained, or removal of it, if it were, would provide a subsequence of $\{a_k\}$ to establish that $|\sum \bar{a}_j| < \max_S |\sum a_j'|$, contrary to the definition of $\{\bar{a}_j\}$.

THEOREM 4. $\rho = 1/\pi$.

PROOF. First,

$$\int_0^{2\pi} F(\phi)d\phi = 2\sum A_k.$$

Thus $\max F(\phi) \geqq \sum A_k/\pi$, whence, by Theorem 1, $\rho \geqq 1/\pi$. To show that $\rho \leqq 1/\pi$, consider the sequence over ν of particular finite sequences $\{{}_\nu a_k\}$, where ${}_\nu a_k \equiv \exp\{k\pi i/(2\nu+1)\}$, $(k = -2\nu, -2\nu+1, \cdots, 0, 1, \cdots, 2\nu, 2\nu+1)$. By Corollary 2.1, for given ν any subsequence $\{{}_\nu a_j'\}$ of $\{{}_\nu a_k\}$ the sum of whose elements is of maximum modulus consists of those elements whose arguments lie in a certain sector of aperture π. By the symmetry of the sequence $\{{}_\nu a_k\}$, the midray of such a sector must lie either on a vector ${}_\nu a_k$ or midway between two such vectors which are adjacent. In the latter case, however, Theorem 3 would be violated. Hence the former must obtain, and thus those elements of $\{{}_\nu a_k\}$ for which $-\pi/2 < k\pi/(2\nu+1) < \pi/2$ constitute a subsequence the sum of whose elements is of maximum modulus. Hence, if $S(\nu)$ denotes the general subsequence $\{{}_\nu a_j'\}$ of $\{{}_\nu a_k\}$,

$$\max_{S(\nu)} \Big|\sum_j {}_\nu a_j'\Big| \Big/ \sum_k {}_\nu A_k = \sum_{k=-\nu}^{\nu} \cos\{k\pi/(2\nu+1)\}/\{2(2\nu+1)\}$$
$$= 1/\{2(2\nu+1)\sin[\pi/\{2(2\nu+1)\}]\};$$

and, as $\nu \to \infty$, this tends monotonely to $1/\pi$. This establishes (ii).

THEOREM 5. *There exists no sequence* $\{a_k\} \in \mathfrak{A}$ *for which* $F(\phi)$ *is constant.*

PROOF. If there were such a sequence $\{a_k\}$ then, by Theorem 1, for each ϕ the sequence $\{a_j^*\}$ of those elements of $\{a_k\}$ for which $\cos(\phi - \phi_k) > 0$ would be such that $|\sum a_j^*| = \max_S |\sum a_j'|$. Hence, by Corollary 1.1 and Theorem 3, there would exist no non-zero element of $\{a_k\}$, contrary to the definition of $\mathfrak{A}$.

THEOREM 6. *Given an arbitrary sequence, finite or infinite, of pairs* (C_k, ψ_k), *where the* ψ_k *are real numbers and the* C_k *positive numbers with* $\sum C_k < \infty$, *then* $\Phi(\phi) \equiv \sum C_k |\cos(\phi - \psi_k)|$ *is not constant.*

PROOF. The sequence $\{a_k\}$ defined thus: $a_{2k-1} \equiv C_k \exp(i\psi_k)$, $a_{2k} \equiv C_k \exp[i(\psi_k - \pi)]$, is of class $\mathfrak{A}$, and

$$F(\phi) = \sum_{\cos(\phi-\psi_k)>0} C_k \cos(\phi - \psi_k) + \sum_{\cos(\phi-\psi_k)<0} C_k \cos(\phi + \pi - \psi_k)$$
$$= \sum C_k |\cos(\phi - \psi_k)| = \Phi(\phi).$$

The conclusion now follows from Theorem 5.

THEOREM 7. *There exists no sequence* $\{a_k\} \epsilon \mathfrak{A}$ *for which it is true that* $\max_S |\sum a_j'| / \sum A_k = \rho$.

PROOF. If there were such a sequence $\{a_k\}$, then, by Theorem 1, $F(\phi) \leqq \rho \sum A_k$ for all ϕ. Hence

$$\int_0^{2\pi} |\rho \sum A_k - F(\phi)| \, d\phi = \int_0^{2\pi} \{\rho \sum A_k - F(\phi)\} d\phi$$
$$= 2\sum A_k - 2\sum A_k = 0.$$

By continuity, then, $F(\phi) = \rho \sum A_k$ for all ϕ. But by Theorem 5 this is impossible. This establishes (iii).

LEMMA 8.1. *Let* X *be an aggregate of elements* x *of any sort, and* $\{f_N\}$ *any sequence of functionals over* X. *Then* $\sup_x \limsup_N f_N(x) \leqq \limsup_N \sup_x f_N(x)$.

PROOF. For each N and for all x, $f_N(x) \leqq \sup_x f_N(x)$. Hence, for all x, $\limsup_N f_N(x) \leqq \limsup_N \sup_x f_N(x)$, and the conclusion follows.

REMARK. Equality in the conclusion of Lemma 8.1 is not implied by the hypotheses. For, if we let X represent the totality of real numbers and define $f_N(1/N) = 1$, $f_N(x) = 0$ for $x \neq 1/N$, $(N = 1, 2, \cdots)$, it follows that $\limsup_N f_N(x) = 0$ for each x, so that $\sup_x \limsup_N f_N(x) = 0$, whereas $\sup_x f_N(x) = 1$ for each N, so that $\limsup_N \sup_x f_N(x) = 1$.

THEOREM 8. *Let* $\{b_k\} \epsilon \mathfrak{B}$ *be arbitrary. Then there exists a subsequence* $\{b_j^*\}$ *of* $\{b_k\}$ *for which*

$$\limsup_N |\sum' b_j^*| / \sum_1^N B_k = \sup_T \limsup_N |\sum' b_j'| / \sum_1^N B_k$$
$$= \limsup_N \sup_T |\sum' b_j'| / \sum_1^N B_k = \limsup_N \max_T |\sum' b_j'| / \sum_1^N B_k.$$

PROOF. By Theorem 1, for each N there exists a subsequence $\{b_j^{(N)}\}$ of $\{b_k\}$ for which $|\sum' b_j^{(N)}| / \sum_1^N B_k = \sup_T |\sum' b_j'| / \sum_1^N B_k$. Let $\{N(\nu)\}$, $(\nu = 1, 2, \cdots)$, be a subsequence of $\{N\}$ such that

$$\lim_{\nu} \left| \sum_1^{'} {}_{\nu}b_j \right| / \sum_1^{N(\nu)} B_k = \limsup_N \left| \sum_1^{'} b_j^{(N)} \right| / \sum_1^{N} B_k,$$

and such that

$$\sum_1^{N(\nu-1)} B_k / \sum_1^{N(\nu)} B_k < 1/2^{\nu+1},$$

where the notation ${}_{\nu}b_j$ represents $b_j^{(N)}$ with $N = N(\nu)$. Define the subsequence $\{b_j^*\}$ of $\{b_k\}$ in such a manner that its elements coincide in order with those of $\{{}_{\nu}b_j\}$ in the subscript interval (with respect to the original sequence $\{b_k\}$) $N(\nu-1) < k \leqq N(\nu)$ for all ν, $(N(0) \equiv 0)$. Now

$$\limsup_N \left| \sum{}' b_j^* \right| / \sum_1^N B_k \geqq \limsup_{\nu} \left| \sum{}' b_j^* \right| / \sum_1^{N(\nu)} B_k,$$

so that from the inequality

$$\left| \sum{}' b_j^* \right| / \sum_1^{N(\nu)} B_k \geqq \left| \sum_1^{'} {}_{\nu}b_j \right| / \sum_1^{N(\nu)} B_k - 1/2^{\nu},$$

it follows that

$$\limsup_N \left| \sum{}' b_j^* \right| / \sum_1^N B_k \geqq \limsup_N \max_T \left| \sum{}' b_j' \right| / \sum_1^N B_k.$$

But that

$$\limsup_N \left| \sum{}' b_j^* \right| / \sum_1^N B_k \leqq \sup_T \limsup_N \left| \sum{}' b_j' \right| / \sum_1^N B_k$$

is obvious, and that

$$\sup_T \limsup_N \left| \sum{}' b_j' \right| / \sum_1^N B_k \leqq \limsup_N \max_T \left| \sum{}' b_j' \right| / \sum_1^N B_k$$

follows from Lemma 8.1. The conclusion follows. This establishes (iv).

LEMMA 9.1. $\sigma \geqq \rho$.

PROOF. By Theorem 8,

$$\sigma \equiv \inf_B \max_T \limsup_N \left| \sum{}' b_j' \right| / \sum_1^N B_k$$

$$= \inf_B \limsup_N \max_T \left| \sum{}' b_j' \right| / \sum_1^N B_k \geqq \rho,$$

which establishes the lemma.

Consider now the sequence $\{b_k^*\}$ defined thus: $b_k^* \equiv e^{ik}$, $(k=1, 2, \cdots)$, and define

$$F_N(\phi) \equiv \sum_1^N \{\cos(\phi - k) + |\cos(\phi - k)|\}/2N, \qquad 0 \leqq \phi \leqq 2\pi.$$

LEMMA 9.2. $\operatorname{Lim}_N \operatorname{osc}_\phi F_N(\phi) = 0$.

PROOF. Let $\epsilon > 0$ be arbitrary; let K be such that, for each ϕ, $\phi \equiv p(\phi) + \eta_\phi \pmod{2\pi}$ for some η_ϕ for which $|\eta_\phi| < \epsilon$ and some integer $p(\phi)$ for which $0 \leqq p(\phi) \leqq K$; and let N be such that $K/N < \epsilon$. Then, for each ϕ,

$$\begin{aligned} |F_N(\phi) - F_N(0)| \leqq & \left| \sum_{k=1}^{N-p(\phi)} \{\cos(k - \eta_\phi) \right. \\ & \left. + |\cos(k - \eta_\phi) - \cos k - |\cos k|\}\right| /2N \\ & + \left| \sum_{k=1}^{p(\phi)} \{\cos(\phi - k) + |\cos(\phi - k)|\}\right| /2N \\ & + \left| \sum_{N-p(\phi)+1}^{N} \{\cos k + |\cos k|\}\right| /2N < 3\epsilon. \end{aligned}$$

This establishes the lemma.

LEMMA 9.3. $\operatorname{Lim}_N F_N(\phi) = \rho$ *uniformly in* ϕ.

PROOF. The assertion follows from Lemma 9.2 and the fact that, for each N, $\int_0^{2\pi} F_N(\phi)\,d\phi = 2$.

THEOREM 9. $\sigma = \rho$.

PROOF. Applying Theorem 2 to the (finite) sequence of those elements of $\{b_k^*\}$ for which $k \leqq N$, we find that

$$\max_T \left| \sum{}' b_j^{*\prime} \right| / \sum_1^N B_k^* = \max_\phi F_N(\phi),$$

which tends to ρ, by Lemma 9.3. By Theorem 8 and Lemma 9.1, this establishes the theorem, and hence also (v).

THEOREM 10. *There exist an uncountably infinite number of subsequences* $\{b_j^*\}$ *of* $\{b_k^*\}$ *for which*

$$\lim_N \left| \sum{}' b_j^* \right| / \sum_1^N B_k^* = \max_T \limsup_N \left| \sum{}' b_j^{*\prime} \right| / \sum_1^N B_k^* = \rho = \sigma.$$

PROOF. Let ϕ' be arbitrary, and let $\{b_i^*\}$ be the sequence of those elements of $\{b_k^*\}$ for which $\cos(\phi'-\phi_k^*)>0$. Then, by inequalities like those used in the proof of Theorem 2, for each N,

$$F_N(\phi') \leqq |\sum{}' b_i^*| / \sum_1^N B_k^* \leqq \max_T |\sum{}' b_i^{*\prime}| / \sum_1^N B_k^* = \max_\phi F_N(\phi),$$

and the conclusion is seen to follow from Lemma 9.3 and Theorem 8. This establishes (vi).

THEOREM 11. *If* $\Phi_N(\phi) \equiv \sum_1^N |\cos(\phi-k)|/N$, $(0 \leqq \phi \leqq 2\pi)$, *then* $\lim_N \Phi_N(\phi) = 2/\pi$ *uniformly in* ϕ.

PROOF. As in the proof of Lemma 9.2, it can be shown that $\lim_N \cos_\phi \Phi_N(\phi) = 0$. Also,

$$\int_0^{2\pi} \Phi_N(\phi) d\phi = 4.$$

The conclusion follows.

REMARK. The sequence $\{b_k^*\}$ could equally well have been taken thus: $b_k^* = e^{i\delta k}$, $(k=1, 2, \cdots)$, where δ is any number incommensurable with π.

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